Power series solutions of ordinary differential equations

Frames 1 to 128

Learning outcomes

When you have completed this Programme you will be able to:

- Obtain the nth derivative of the exponential and circular and hyperbolic functions
- Apply the Leibnitz theorem to derive the nth derivative of a product of expressions
- Use the Leibnitz–Maclaurin method of obtaining a series solution to a second-order homogeneous differential equation with constant coefficients
- Use Frobenius' method of obtaining a series solution to a second-order homogeneous differential equation for different cases of the indicial equation
- Apply Frobenius' method to Bessel's equation to derive Bessel functions of the first kind
- Apply Frobenius' method to Legendre's equation to derive Legendre polynomials
- Use Rodrigue's formula to derive Legendre polynomials and the generating function to obtain some of their properties
- Recognise a Sturm–Liouville system and the orthogonality properties of its eigenfunctions
- Write a polynomial in x as a finite series of Legendre polynomials

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 13 Series 1, 14 Series 2 and 25 Second-order differential equations

Higher derivatives

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If $y = \sin x$ $\quad \dfrac{dy}{dx} = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$

$$\frac{d^2y}{dx^2} = -\sin x = \sin(x + \pi) = \sin\left(x + \frac{2\pi}{2}\right)$$

$$\frac{d^3y}{dx^3} = -\cos x = \sin\left(x + \frac{3\pi}{2}\right) \quad \text{etc.}$$

We see a pattern developing. In general $\dfrac{d^ny}{dx^n} = \sin\left(x + \dfrac{n\pi}{2}\right)$. Before we go further, we introduce a shorthand notation for the nth derivative of y as $y^{(n)} = \dfrac{d^ny}{dx^n}$. Note, however, we still use the 'prime' notation y', y'' and y''' to represent the first, second and third derivatives respectively.
The results above can therefore be written

$$\text{If } y = \sin x \quad \therefore\ y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$y'' = -\sin x = \sin\left(x + \frac{2\pi}{2}\right)$$

$$y''' = -\cos x = \sin\left(x + \frac{3\pi}{2}\right)$$

and, in general, $y^{(n)} = \sin\left(x + \dfrac{n\pi}{2}\right)$

It is therefore possible to write down any particular derivative of $\sin x$ without calculating all the previous derivatives. For example

$$\frac{d^7y}{dx^7} = y^{(7)} = \sin\left(x + \frac{7\pi}{2}\right) = -\cos x$$

Similarly, starting with $y = \cos x$, we can determine an expression for the nth derivative of y which is ....

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$$\boxed{y^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)}$$

Because

$$y = \cos x \quad \therefore\ y' = -\sin x = \cos\left(x + \frac{\pi}{2}\right)$$

$$y'' = -\cos x = \cos\left(x + \frac{2\pi}{2}\right)$$

$$y''' = \sin x \quad = \cos\left(x + \frac{3\pi}{2}\right) \quad \text{etc.}$$

$$\therefore\ y^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$

Many of the standard functions can be treated in a similar manner. For example, if $y = e^{ax}$, then $y^{(n)} =$ ....

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$$\boxed{y^{(n)} = a^n e^{ax}}$$

Because

$$y = e^{ax}, \quad y' = ae^{ax}, \quad y'' = a^2e^{ax}, \quad y''' = a^3e^{ax}, \quad \text{etc.}$$

In general, $y^{(n)} = a^n e^{ax}$.

With no great effort, we can now write down expressions for the following

If $y = \sin ax$, $y^{(n)} = \ldots\ldots\ldots\ldots$

If $y = \cos ax$, $y^{(n)} = \ldots\ldots\ldots\ldots$

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$$y = \sin ax, \quad y^{(n)} = a^n \sin\left(ax + \frac{n\pi}{2}\right)$$
$$y = \cos ax, \quad y^{(n)} = a^n \cos\left(ax + \frac{n\pi}{2}\right)$$

Now one more.

If $y = \ln x$, $y^{(n)} = \ldots\ldots\ldots\ldots$

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$$\boxed{y^{(n)} = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n}}$$

Because

$$y = \ln x \quad \therefore \; y' = \frac{1}{x}$$
$$y'' = -\frac{1}{x^2}$$
$$y''' = \frac{2}{x^3}$$
$$y^{(4)} = -\frac{3!}{x^4} \qquad \therefore \; y^{(n)} = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n}$$

We already know that, if $y = \ln x$, $\dfrac{dy}{dx} = y' = \dfrac{1}{x} = x^{-1}$.

Therefore, if the result obtained for $y^{(n)}$ is to be valid for $n = 1$, then

$$y' = (-1)^0 \cdot \frac{0!}{x} = \frac{0!}{x}$$

But $y' = x^{-1}$ $\qquad \therefore \; 0! = \ldots\ldots\ldots\ldots$

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$$\boxed{0! = 1}$$

Now let us consider the derivatives of $\sinh ax$ and $\cosh ax$.

Next frame

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If $y = \sinh ax$, $\quad y' = a\cosh ax$

$$y'' = a^2 \sinh ax$$
$$y''' = a^3 \cosh ax \qquad \text{etc.}$$

Because $\sinh ax$ is not periodic, we cannot proceed as we did with $\sin ax$. We need to find a general statement for $y^{(n)}$ containing terms in $\sinh ax$ and in $\cosh ax$, such that, when n is even, the term in $\cosh ax$ disappears and, when n is odd, the term in $\sinh ax$ disappears.

This we can do by writing $y^{(n)}$ in the form

$$y^{(n)} = \frac{a^n}{2}\{[1 + (-1)^n]\sinh ax + [1 - (-1)^n]\cosh ax\}$$

In very much the same way, we can determine the nth derivative of $y = \cosh ax$ as ....

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$$\boxed{y^{(n)} = \frac{a^n}{2}\{[1 - (-1)^n]\sinh ax + [1 + (-1)^n]\cosh ax\}}$$

Finally, let us deal with $y = x^a$.

$$y = x^a \quad \therefore \quad y' = ax^{a-1}$$
$$y'' = a(a-1)\,x^{a-2}$$
$$y''' = a(a-1)(a-2)\,x^{a-3}$$
$$\dots\dots\dots\dots$$
$$\therefore \quad y^{(n)} = a(a-1)(a-2)\dots(a-n+1)\,x^{a-n}$$
$$\therefore \quad y^{(n)} = \frac{a!}{(a-n)!}\,x^{a-n} \qquad (a \text{ is a positive integer})$$

So, collecting our results together, we have

$$y = x^a \qquad y^{(n)} = \frac{a!}{(a-n)!}x^{a-n}$$
$$y = e^{ax} \qquad y^{(n)} = a^n e^{ax}$$
$$y = \sin ax \qquad y^{(n)} = a^n \sin\left(ax + \frac{n\pi}{2}\right)$$
$$y = \cos ax \qquad y^{(n)} = a^n \cos\left(ax + \frac{n\pi}{2}\right)$$
$$y = \sinh ax \qquad y^{(n)} = \frac{a^n}{2}\{[1 + (-1)^n]\sinh ax + [1 - (-1)^n]\cosh ax\}$$
$$y = \cosh ax \qquad y^{(n)} = \frac{a^n}{2}\{[1 - (-1)^n]\sinh ax + [1 + (-1)^n]\cosh ax\}$$

Make a note of these, as a set, and then move on to the next frame

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Exercise

Determine the following derivatives

1 $y = \sin 4x \qquad y^{(5)} = \ldots\ldots\ldots\ldots$

2 $y = e^{x/2} \qquad y^{(8)} = \ldots\ldots\ldots\ldots$

3 $y = \cosh 3x \qquad y^{(12)} = \ldots\ldots\ldots\ldots$

4 $y = \cos(x\sqrt{2}) \qquad y^{(10)} = \ldots\ldots\ldots\ldots$

5 $y = x^8 \qquad y^{(6)} = \ldots\ldots\ldots\ldots$

6 $y = \sinh 2x \qquad y^{(7)} = \ldots\ldots\ldots\ldots$

Finish them all; then check with the next frame

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Here are the solutions

1 $y^{(5)} = 4^5 \sin\left(4x + \frac{5\pi}{2}\right) = 1024 \sin\left(4x + \frac{\pi}{2}\right) = 1024 \cos 4x$

2 $y^{(8)} = \left(\frac{1}{2}\right)^8 e^{x/2} = \frac{1}{256} e^{x/2} = e^{x/2}/256$

3 $y^{(12)} = \frac{3^{12}}{2}\{0 \sinh 3x + 2\cosh 3x\} = 3^{12} \cosh 3x$

4 $y^{(10)} = (\sqrt{2})^{10} \cos\left(x\sqrt{2} + \frac{10\pi}{2}\right)$

$= 32\cos(x\sqrt{2} + 5\pi) = -32\cos(x\sqrt{2})$

5 $y^{(6)} = \frac{8!}{2!} x^2 = 20\,160\, x^2$

6 $y^{(7)} = \frac{2^7}{2}\left\{[1 + (-1)^7]\sinh 2x + [1 - (-1)^7]\cosh 2x\right\}$

$= 2^7 \cosh 2x$

Leibnitz theorem – *n*th derivative of a product of two functions

If $y = uv$, where u and v are functions of x, then

$$y' = uv' + vu' \quad \text{where} \quad v' = \frac{dv}{dx} \quad \text{and} \quad u' = \frac{du}{dx}$$

$$\text{and} \quad y'' = uv'' + v'u' + vu'' + u'v' = u''v + 2u'v' + uv''$$

If we differentiate the last result and collect like terms, we obtain

$$y''' = \ldots\ldots\ldots\ldots$$

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$$\boxed{y''' = u'''v + 3u''v' + 3u'v'' + uv'''}$$

A further stage of differentiation would give

$$y^{(4)} = u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}$$

These results can therefore be written

$$\begin{aligned}
y &= uv \\
y' &= u'v + uv' \\
y'' &= u''v + 2u'v' + uv'' \\
y''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\
y^{(4)} &= u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}
\end{aligned}$$

Notice that in each case

(a) the superscript of u decreases regularly by 1

(b) the superscript of v increases regularly by 1

(c) the numerical coefficients are the normal binomial coefficients.

Indeed, $(uv)^{(n)}$ can be obtained by expanding $(u+v)^{(n)}$ using the binomial theorem where the 'powers' are interpreted as derivatives. So the expression for the nth derivative can therefore be written as

$$\begin{aligned}
y^{(n)} &= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{1 \times 2}u^{(n-2)}v^{(2)} \\
&\quad + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}u^{(n-3)}v^{(3)} + \ldots \\
&= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} \\
&\quad + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \ldots
\end{aligned}$$

i.e. $$\begin{aligned}
y^{(n)} &= u^{(n)}v + {}^nC_1\, u^{(n-1)}v^{(1)} + {}^nC_2 u^{(n-2)}v^{(2)} + \ldots \\
&\quad + {}^nC_{n-1}u^{(1)}v^{(n-1)} + uv^{(n)}
\end{aligned}$$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

If $y = uv$ $$y^{(n)} = \sum_{r=0}^{n} {}^nC_r\, u^{(n-r)}v^{(r)} \quad \text{where } u^{(0)} \equiv u$$

This is the *Leibnitz theorem*. We shall certainly be using it often in the work ahead, so make a note of it for future reference. Then we can see it in use.

Choice of function for *u* and *v*

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For the product $y = uv$ the function taken as

(a) u is the one whose nth derivative can readily be obtained
(b) v is the one whose derivatives reduce to zero after a small number of stages of differentiation.

Example 1

To find $y^{(n)}$ when $y = x^3 e^{2x}$.

Here we choose $v = x^3$ – whose fourth derivative is zero

$u = e^{2x}$ – because we know that the nth derivative

$u^{(n)} = \dots\dots\dots\dots$

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$$\boxed{u^{(n)} = 2^n e^{2x}}$$

Using the Leibnitz theorem:

$$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \dots$$

$v = x^3$; $v^{(1)} = 3x^2$; $v^{(2)} = 6x$; $v^{(3)} = 6$; $v^{(4)} = 0$

$u = e^{2x}$; $u^{(n)} = 2^n e^{2x}$

$\therefore\ y^{(n)} = \dots\dots\dots\dots$

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$$\boxed{y^{(n)} = e^{2x}\, 2^{n-3}\{8x^3 + 12nx^2 + n(n-1)\,6x + n(n-1)(n-2)\}}$$

Example 2

If $x^2y'' + xy' + y = 0$, show that

$$x^2y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2+1)\,y^{(n)} = 0.$$

We take the given equation $x^2y'' + xy' + y = 0$ and differentiate n times, treating each term in turn.

If $w = x^2y''$ $\quad w^{(n)} = \dots\dots\dots\dots$

If $w = xy'$ $\quad w^{(n)} = \dots\dots\dots\dots$

If $w = y$ $\quad w^{(n)} = \dots\dots\dots\dots$

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$$\begin{array}{|lll|}\hline w = x^2y'' & \therefore & w^{(n)} = y^{(n+2)}\,x^2 + ny^{(n+1)}\,2x + \dfrac{n(n-1)}{2!}\,y^{(n)}2 + 0\ldots \\ w = xy' & \therefore & w^{(n)} = y^{(n+1)}\,x + ny^{(n)}1 + 0 + \ldots \\ w = y & \therefore & w^{(n)} = y^{(n)} \\ \hline\end{array}$$

Then $\left[x^2y'' + xy' + y\right]^{(n)} = 0$ becomes

....

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$$\boxed{x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0}$$

which is what we had to show.

Example 3

Differentiate n times

$$(1+x^2)y'' + 2xy' - 5y = 0.$$

The result ....

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$$\boxed{(1+x^2)\,y^{(n+2)} + 2(n+1)\,xy^{(n+1)} + (n^2+n-5)\,y^{(n)} = 0}$$

Because, by the Leibnitz theorem

$$\left\{y^{(n+2)}(1+x^2) + ny^{(n+1)}\,2x + \frac{n(n-1)}{2!}\,y^{(n)}2\right\}$$
$$+ 2\left\{xy^{(n+1)} + ny^{(n)}\cdot 1\right\} - 5y^{(n)} = 0$$
$$(1+x^2)\,y^{(n+2)} + 2(n+1)\,xy^{(n+1)} + \{n(n-1) + 2n - 5\}\,y^{(n)} = 0$$
$$(1+x^2)\,y^{(n+2)} + 2(n+1)\,xy^{(n+1)} + (n^2+n-5)\,y^{(n)} = 0$$

We shall be using the Leibnitz theorem in the rest of this Programme, so let us move on to see some of its applications.

Power series solutions

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Second-order linear differential equations with constant coefficients of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$ can be solved by algebraic methods giving solutions in terms of the normal elementary functions such as exponentials, trigonometric and polynomial functions.

▶

In general, equations of the form $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$, where $P(x)$ and $Q(x)$ are functions of x, cannot be solved in this way. However, it is often possible to obtain solutions in the form of infinite series of powers of x – and the next section of work investigates some of the methods which make this possible.

1 Leibnitz–Maclaurin method

As the title suggests, for this we need to be familiar with the Leibnitz theorem and with Maclaurin's series.

The *Leibnitz theorem* states that, if $y = uv$, where u and v are functions of x, then

$$y^{(n)} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} y^{(n)} &= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \ldots \\ &\quad + \frac{n(n-1)\ldots(n-r+1)}{r!}u^{(n-r)}v^{(r)} + \ldots + uv^{(n)} \end{aligned}}$$

where $u^{(r)}$ and $v^{(r)}$ denote $\dfrac{d^r u}{dx^r}$ and $\dfrac{d^r v}{dx^r}$ respectively.

Maclaurin's series for $y = f(x)$ can be stated as

$$y = \ldots\ldots\ldots\ldots$$

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$$\boxed{y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \ldots + \frac{x^n}{n!}\left(y^{(n)}\right)_0 + \ldots}$$

where $\left(y^{(n)}\right)_0$ denotes the value of the nth derivative of y at $x = 0$.

On to the next frame

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Example 1

Find the power series solution of the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 1.$$

The equation can be written

$$xy'' + y' + xy = 1$$

In the first product term xy'', treat y'' as u and x as v. Then, differentiating the equation n times by the Leibnitz theorem, gives

$$\ldots\ldots\ldots\ldots$$

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$$\left(xy^{(n+2)} + n \cdot 1 \cdot y^{(n+1)}\right) + y^{(n+1)} + \left(xy^{(n)} + n \cdot 1 \cdot y^{(n-1)}\right) = 0$$
$$\text{i.e.} \quad xy^{(n+2)} + (n+1)y^{(n+1)} + xy^{(n)} + ny^{(n-1)} = 0$$

At $x = 0$, this becomes

$$(n+1)\left(y^{(n+1)}\right)_0 + n\left(y^{(n-1)}\right)_0 = 0$$

$$\therefore \left(y^{(n+1)}\right)_0 = -\frac{n}{n+1}\left(y^{(n-1)}\right)_0 \qquad n \geq 1$$

This relationship is called a *recurrence relation.*

We can now substitute $n = 1, 2, 3, \ldots$ and get a set of relationships between the various coefficients.

$n = 1 \quad (y'')_0 = -\frac{1}{2}(y)_0$

$n = 2 \quad (y''')_0 = -\frac{2}{3}(y')_0$

$n = 3 \quad (y^{(4)})_0 = -\frac{3}{4}(y'')_0 = (-\frac{3}{4})(-\frac{1}{2})(y)_0$

Continuing in the same way,

$(y^{(5)})_0 = \ldots\ldots\ldots\ldots$

$(y^{(6)})_0 = \ldots\ldots\ldots\ldots$

$(y^{(7)})_0 = \ldots\ldots\ldots\ldots$

$(y^{(8)})_0 = \ldots\ldots\ldots\ldots$

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$n = 4$	$(y^{(5)})_0 = -\frac{4}{5}(y^{(3)})_0 = (-\frac{4}{5})(-\frac{2}{3})(y^{(1)})_0$
$n = 5$	$(y^{(6)})_0 = -\frac{5}{6}(y^{(4)})_0 = (-\frac{5}{6})(-\frac{3}{4})(-\frac{1}{2})(y)_0$
$n = 6$	$(y^{(7)})_0 = -\frac{6}{7}(y^{(5)})_0 = (-\frac{6}{7})(-\frac{4}{5})(-\frac{2}{3})(y^{(1)})_0$
$n = 7$	$(y^{(8)})_0 = -\frac{7}{8}(y^{(6)})_0 = (-\frac{7}{8})(-\frac{5}{6})(-\frac{3}{4})(-\frac{1}{2})(y)_0$

Notice that, by this means, the values of all the derivatives at $x = 0$ can be expressed in terms of $(y)_0$ and $(y')_0$.

If we now substitute these values for $(y^{(r)})_0$ in the Maclaurin series

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \ldots + \frac{x^r}{r!}(y^{(r)})_0 + \ldots$$

we obtain $\ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned} y =&(y)_0 + x(y')_0 + \frac{x^2}{2!}\left(-\frac{1}{2}\right)(y)_0 + \frac{x^3}{3!}\left(-\frac{2}{3}\right)(y')_0 \\ &+\frac{x^4}{4!}\left(-\frac{3}{4}\right)\left(-\frac{1}{2}\right)(y)_0 + \frac{x^5}{5!}\left(-\frac{4}{5}\right)\left(-\frac{2}{3}\right)(y')_0 \\ &+\frac{x^6}{6!}\left(-\frac{5}{6}\right)\left(-\frac{3}{4}\right)\left(-\frac{1}{2}\right)(y)_0 + \dots\dots\dots\dots \end{aligned}}$$

Simplifying, this gives

$$y = (y)_0\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} - \frac{x^6}{2^2 \times 4^2 \times 6^2} + \dots\right\}$$
$$+ (y')_0\left\{x - \frac{x^3}{3^2} + \frac{x^5}{3^2 \times 5^2} + \dots\right\}$$

The values of $(y)_0$ and $(y')_0$ provide the two arbitrary constants for the second-order equation and are obtained from the given initial conditions.

For example, if at $x = 0$, $y = 2$ and $\dfrac{dy}{dx} = 1$, then the relevant particular solution is ....

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$$\boxed{\begin{aligned} y = 2&\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} - \frac{x^6}{2^2 \times 4^2 \times 6^2} + \dots\right\} \\ &+\left\{x - \frac{x^3}{3^2} + \frac{x^5}{3^2 \times 5^2} + \dots\right\} \end{aligned}}$$

Because at $x = 0$, $y = 2$ i.e. $(y)_0 = 2$

$\dfrac{dy}{dx} = 1$ i.e. $(y')_0 = 1$.

To be a valid solution, the series obtained must converge. Application of the ratio test will normally indicate any restrictions on the values that x may have.

The Leibnitz–Maclaurin (power series) method therefore involves the following main steps:

(a) Differentiate the given equation n times, using the Leibnitz theorem.

(b) Rearrange the result to obtain the recurrence relation at $x = 0$.

(c) Determine the values of the derivatives at $x = 0$, usually in terms of $(y)_0$ and $(y')_0$.

(d) Substitute in the Maclaurin expansion for $y = f(x)$.

(e) Simplify the result where possible and apply boundary conditions if provided.

That is all there is to it. Let us go through the various steps with another example. ▶

Example 2

Determine a series solution of the equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

The equation can be written $y''+xy'+y=0$

(a) Differentiate n times using the Leibnitz theorem, which gives

....

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$$\boxed{y^{(n+2)}+xy^{(n+1)}+(n+1)y^{(n)}=0}$$

Because $y''+xy'+y=0$

$$\therefore\; y^{(n+2)}+\left\{xy^{(n+1)}+n\cdot 1\cdot y^{(n)}\right\}+y^{(n)}=0$$

$$\therefore\; y^{(n+2)}+xy^{(n+1)}+(n+1)y^{(n)}=0.$$

(b) Determine the recurrence relation at $x=0$, which is

....

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$$\boxed{y^{(n+2)}=-(n+1)\,y^{(n)}}$$

(c) Now taking $n=0$, 1, 2, 3, 4, 5, determine the derivatives at $x=0$ in terms of $(y)_0$ and $(y')_0$. List them, as we did before, in table form.

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$n=0$	$(y'')_0=-(y)_0$	$=-(y)_0$
1	$(y''')_0=-2\,(y')_0$	$=-2\,(y')_0$
2	$(y^{(4)})_0=-3\,(y'')_0=(-3)[-(y)_0]$	$=3\,(y)_0$
3	$(y^{(5)})_0=-4\,(y''')_0=(-4)[-2(y')_0]$	$=2\times 4\,(y')_0$
4	$(y^{(6)})_0=-5\,(y^{(4)})_0=(-5)[-3(y'')_0]$	$=-3\times 5\,(y)_0$
5	$(y^{(7)})_0=-6\,(y^{(5)})_0=(-6)[-4(y''')_0]$	$=-2\times 4\times 6\,(y')_0$

(d) Substitute these expressions for the derivatives in terms of $(y)_0$ and $(y')_0$ in Maclaurin's expansion

$$y=(y)_0+x\,(y')_0+\frac{x^2}{2!}(y'')_0+\frac{x^3}{3!}(y''')_0+\frac{x^4}{4!}(y^{(4)})_0+\dots$$

Then $y=$ ....

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$$\boxed{\begin{aligned}y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(-y)_0 + \frac{x^3}{3!}(-2y')_0 + \frac{x^4}{4!}(3y)_0 + \frac{x^5}{5!}(8y')_0\\ + \frac{x^6}{6!}(-15y)_0 + \frac{x^7}{7!}(-48y')_0 + \dots\end{aligned}}$$

Collecting now the terms in $(y)_0$ and $(y')_0$, we finally obtain

$$y = (y)_0\left\{1 - \frac{x^2}{2} + \frac{x^4}{2 \times 4} - \frac{x^6}{2 \times 4 \times 6} + \dots\right\}$$
$$+ (y')_0\left\{x - \frac{x^3}{3} + \frac{x^5}{3 \times 5} - \frac{x^7}{3 \times 5 \times 7} + \dots\right\}$$

They are all done in very much the same way. Here is another.

Example 3

Solve the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 2xy = 0$ given that at $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 1$.

First write the equation as $y'' + y' + 2xy = 0$, differentiate n times by the Leibnitz theorem and obtain the recurrence relation at $x = 0$, which is ....

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$$\boxed{y^{(n+2)} = -\left\{y^{(n+1)} + 2ny^{(n-1)}\right\} \quad n \geq 1}$$

Because $y'' + y' + 2xy = 0$

$$\therefore\ y^{(n+2)} + y^{(n+1)} + 2xy^{(n)} + n2y^{(n-1)} = 0$$

At $x = 0$, $\quad y^{(n+2)} + y^{(n+1)} + 2ny^{(n-1)} = 0$

$$\therefore\ y^{(n+2)} = -\left\{y^{(n+1)} + 2ny^{(n-1)}\right\}$$

Since we have a term in $y^{(n-1)}$, then n must start at 1 to give $(y)_0$. Therefore the recurrence relation applies for $n \geq 1$.

We now take $n = 1, 2, 3, \dots$ to obtain the relationships between the coefficients up to $(y^{(6)})_0$. Complete the table and check with the next frame.

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$$\begin{array}{|lll|}\hline n=1 & (y^{(3)})_0 & =-\{(y^{(2)})_0+2(y)_0\} \\ n=2 & (y^{(4)})_0 & =-\{(y^{(3)})_0+4(y')_0\} \\ n=3 & (y^{(5)})_0 & =-\{(y^{(4)})_0+6(y^{(2)})_0\} \\ n=4 & (y^{(6)})_0 & =-\{(y^{(5)})_0+8(y^{(3)})_0\} \\ \hline\end{array}$$

We therefore have expressions for $(y''')_0, (y^{(4)})_0, (y^{(5)})_0, (y^{(6)})_0$, but what about $(y'')_0$?

If we refer to the initial conditions, we know that at $x=0$, $y=0$ and $y'=1$. $\therefore\ (y)_0=0$ and $(y')_0=1$.

We can find $(y'')_0$ by reference to the given equation itself, because

$$y''+y'+2xy=0$$

Therefore, at $x=0$, $(y'')_0+(y')_0=0\ \therefore\ (y'')_0=-(y')_0=-1$.

So now we have
$$\begin{aligned}(y)_0&=0\\(y')_0&=1\\(y'')_0&=-1\\(y''')_0&=-\{(y'')_0+2(y)_0\}&&=-\{(-1)+0\}&&=1\\(y^{(4)})_0&=-\{(y''')_0+4(y')_0\}&&=-\{1+4\}&&=-5\\(y^{(5)})_0&=-\{(y^{(4)})_0+6(y'')_0\}&&=-\{(-5)-6\}&&=11\\(y^{(6)})_0&=-\{(y^{(5)})_0+8(y''')_0\}&&=-\{11+8\}&&=-19\end{aligned}$$

The required series solution is therefore

$y=$ ....

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$$\boxed{y=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{5x^4}{4!}+\frac{11x^5}{5!}-\frac{19x^6}{6!}+\dots}$$

Because

$$\begin{aligned}y&=(y)_0+x(y')_0+\frac{x^2}{2!}(y'')_0+\frac{x^3}{3!}(y''')_0+\frac{x^4}{4!}(y^{(4)})_0+\dots\\&=0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(1)+\frac{x^4}{4!}(-5)+\frac{x^5}{5!}(11)+\frac{x^6}{6!}(-19)\end{aligned}$$

$$\therefore\ y=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{5x^4}{4!}+\frac{11x^5}{5!}-\frac{19x^6}{6!}+\dots$$

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One more of the same kind.

Example 4

Determine the general series solution of the equation

$$(x^2+1)y''+xy'-4y=0$$

As usual, establish the recurrence relation at $x=0$, which is

....

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$$\boxed{y^{(n+2)} = (4 - n^2)y^{(n)}}$$

Because

$(x^2 + 1)y'' + xy' - 4y = 0$ therefore

$$\left\{(x^2+1)y^{(n+2)} + 2xny^{(n+1)} + 2\frac{n(n-1)}{2!}y^{(n)}\right\} + \left\{xy^{(n+1)} + ny^{(n)}\right\} - 4y^{(n)}$$
$$= 0$$

At $x = 0$, this becomes

$y^{(n+2)} + n(n-1)y^{(n)} + ny^{(n)} - 4y^{(n)} = 0$ that is $y^{(n+2)} = (4 - n^2)y^{(n)}$

Then, starting with $n = 0$, determine expressions for $(y^{(n)})_0$ as far as $n = 7$.

They are ....

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$n = 0$	$(y'')_0 = 4(y)_0$	$= 4(y)_0$
$n = 1$	$(y''')_0 = 3(y')_0$	$= 3(y')_0$
$n = 2$	$(y^{(4)})_0 = 0$	$= 0$
$n = 3$	$(y^{(5)})_0 = -5(y''')_0$	$= -15(y')_0$
$n = 4$	$(y^{(6)})_0 = -12(y^{(4)})_0$	$= 0$
$n = 5$	$(y^{(7)})_0 = -21(y^{(5)})_0$	$= (-21)(-15)(y')_0$

Now substitute in Maclaurin's expansion and simplify the result.

$y =$ ....

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$$\boxed{y = A(1 + 2x^2) + B\left\{x + \frac{x^3}{2} - \frac{x^5}{8} + \frac{x^7}{16} + \dots\right\}}$$

Because

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \frac{x^4}{4!}(y^{(4)})_0 + \dots$$
$$= (y)_0 + x(y')_0 + \frac{x^2}{2!}4(y)_0 + \frac{x^3}{3!}3(y')_0 + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(-15)(y')_0 + \text{etc.}$$
$$= (y)_0\{1 + 2x^2\} + (y')_0\left\{x + \frac{x^3}{2} - \frac{x^5}{8} + \frac{x^7}{16} + \dots\right\}$$

Putting $(y)_0 = A$ and $(y')_0 = B$, we have the result stated.

Now to something slightly different

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2 Frobenius' method

In each of the previous examples, we established the solution as a power series in integral powers of x. Such a solution is not always possible and a more general method is to assume a trial solution of the form

$$y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots\}$$

where a_0 is the first coefficient that is not zero.

The type of equation that can be solved by this method is of the form

$$y'' + Py' + Qy = 0$$

where P and Q are functions of x.

However, certain conditions have to be satisfied.

(a) If the functions P and Q are such that both are finite when x is put equal to zero, $x = 0$ is called an *ordinary point* of the equation.
(b) If xP and x^2Q remain finite at $x = 0$, then $x = 0$ is called a *regular singular point* of the equation.

In both of these cases, the method of Frobenius can be applied.

(c) If, however, P and Q do not satisfy either of these conditions stated in (a) or (b), then $x = 0$ is called an *irregular singular point* of the equation and the method of Frobenius cannot be applied.

Solution of differential equations by the method of Frobenius

To solve a given equation, we have to find the coefficients $a_0, a_1, a_2, \ldots$ and also the index c in the trial solution. Basically, the steps in the method are as follows

(a) Differentiate the trial series as required.
(b) Substitute the results in the given differential equation.
(c) Equate coefficients of corresponding powers of x on each side of the equation.

The following examples will demonstrate the method – so move on

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Example 1

Find a series solution for the equation

$$2x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0.$$

The equation can be written as $2xy''+y'+y=0$.

Assume a solution of the form

$$y=x^c\{a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_rx^r+\ldots\} \qquad a_0\neq 0.$$
$$\therefore\ y=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+\ldots+a_rx^{c+r}+\ldots$$

Differentiating term by term, we get

$$y'=\ldots\ldots\ldots\ldots$$

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$$\boxed{y'=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\ldots+a_r(c+r)x^{c+r-1}+\ldots}$$

Repeating the process one stage further, we have

$y''=\ldots\ldots\ldots\ldots$ (give yourself plenty of room)

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$$\boxed{\begin{aligned}y''=a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}+a_2(c+1)(c+2)x^c+\ldots\\+a_r(c+r-1)(c+r)x^{c+r-2}+\ldots\end{aligned}}$$

So far, we have $\quad 2xy''+y'+y=0$

$$\begin{aligned}y&=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+\ldots+a_rx^{c+r}+\ldots\\
y'&=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\ldots\\
&\quad+a_r(c+r)x^{c+r-1}+\ldots\\
y''&=a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}+a_2(c+1)(c+2)x^c+\ldots\\
&\quad+a_r(c+r-1)(c+r)x^{c+r-2}+\ldots\end{aligned}$$

Considering each term of the equation in turn

$$\begin{aligned}2xy''&=2a_0c(c-1)x^{c-1}+2a_1c(c+1)x^c+2a_2(c+1)(c+2)x^{c+1}\\
&\quad+\ldots+a_r(c+r-1)(c+r)x^{c+r-1}+\ldots\\
y'&=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\ldots\\
&\quad+a_r(c+r)x^{c+r-1}+\ldots\\
y&=a_0x^c+a_1x^{c+1}+\ldots+a_rx^{c+r}+\ldots\end{aligned}$$

Adding these three lines to form the left-hand side of the equation, we can equate the total coefficient of each power of x to zero, since the right-hand side is zero.

$[x^{c-1}]$ gives $\ldots\ldots\ldots\ldots$

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$$\begin{aligned}[x^{c-1}]: \quad & 2a_0c(c-1) + a_0c = 0 \\ & \therefore \; a_0c(2c-1) = 0\end{aligned}$$

So, $[x^{c-1}]$ gives $\quad a_0c(2c-1) = 0 \quad (1)$

Similarly, $[x^c]$ gives ....

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$$2a_1c\,(c+1) + a_1(c+1) + a_0 = 0$$

Simplifying, this becomes

$$a_1(2c^2 + 3c + 1) + a_0 = 0$$

i.e. $\quad a_1(c+1)(2c+1) + a_0 = 0 \quad (2)$

Also $[x^{c+1}]$ gives ....

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$$2a_2(c+1)(c+2) + a_2(c+2) + a_1 = 0$$

and this simplifies straight away to

$a_2(c+2)(2c+3) + a_1 = 0 \quad (3)$

Note that the coefficient of x^c involves all three lines of the expressions and, from then on, a general relationship can be obtained for x^{c+r}, $r \geq 0$.

In the expression for $2xy''$ and y' we have terms in x^{c+r-1}. If we replace r by $(r+1)$, we shall obtain the corresponding terms in x^{c+r}.

In the series for $2xy''$, this is $2a_{r+1}(c+r)(c+r+1)x^{c+r}$
In the series for y', this is $a_{r+1}(c+r+1)x^{c+r}$
In the series for y, this is a_rx^{c+r}

Therefore, equating the total coefficient of x^{c+r} to zero, we have

....

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$$2a_{r+1}(c+r)(c+r+1) + a_{r+1}(c+r+1) + a_r = 0$$

and this tidies up to

$a_{r+1}\{(c+r+1)(2c+2r+1)\} + a_r = 0 \quad (4)$

Make a note of results (1), (2), (3) and (4): we shall return to them in due course.

Then move on

Indicial equation

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Equation (1), formed from the coefficient of the lowest power of x, that is x^{c-1}, is called the *indicial equation* from which the values of c can be obtained. In the present example $a_0c(2c-1)=0$

$$\therefore\ c = \ldots\ldots\ldots\ldots$$

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$$\boxed{c=0 \text{ or } \frac{1}{2}, \text{ since } a_0 \neq 0, \text{ by definition}}$$

Both values of c are valid, so that we have two possible solutions of the given equation. We will consider each in turn.

(a) *Using* $c=0$

(2) gives $a_1(1)(1)+a_0=0 \quad \therefore\ a_1=-a_0$

Similarly

(3) gives $\ldots\ldots\ldots\ldots$

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$$\boxed{a_2(2)(3)+a_1=0}$$

$$a_1=-a_0 \quad \text{and} \quad a_2=-\frac{a_1}{2\times 3}=\frac{a_0}{2\times 3}$$

$$\text{and from (4)} \qquad a_{r+1}=\frac{-a_r}{(r+1)(2r+1)} \qquad r\geq 0$$

From the combined series, the term in x^c and all subsequent terms involve all three lines and the coefficient of the general term can be used.

So we have $a_1=-a_0$ and $a_{r+1}=\dfrac{-a_r}{(r+1)(2r+1)}$ for $r=0,\ 1,\ 2,\ldots$

$$\therefore\ a_2=\frac{-a_1}{2\times 3}=\frac{a_0}{2\times 3}$$

$$a_2=\frac{-a_2}{3\times 5}=\frac{-a_0}{(2\times 3)(3\times 5)}$$

$$a_4=\frac{-a_3}{4\times 7}=\frac{a_0}{(2\times 3\times 4)(3\times 5\times 7)} \qquad \text{etc.}$$

$$\therefore\ y=x^0\left\{a_0-a_0x+\frac{a_0}{(2\times 3)}x^2-\frac{a_0}{(2\times 3)(3\times 5)}x^3+\ldots\right\}$$

$$\therefore\ y=a_0\left\{1-x+\frac{x^2}{(2)(3)}-\frac{x^3}{(2\times 3)(3\times 5)}+\frac{x^4}{(2\times 3\times 4)(3\times 5\times 7)}+\ldots\right\}$$

Now we go through the same steps using our second value for c, i.e. $c=\frac{1}{2}$.

Next frame

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(b) *Using* $c = \frac{1}{2}$

Our equations relating the coefficients were

$$a_0 c(2c-1) = 0 \quad \text{which gave} \quad c = 0 \text{ or } c = \tfrac{1}{2} \qquad (1)$$

$$a_1(c+1)(2c+1) + a_0 = 0 \qquad (2)$$

$$a_2(c+2)(2c+3) + a_1 = 0 \qquad (3)$$

$$a_{r+1}(c+r+1)(2c+2r+1) + a_r = 0 \qquad (4)$$

Putting $c = \frac{1}{2}$ in (2) gives ....

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$$\boxed{a_1 = -\frac{a_0}{3}}$$

Similarly (3) gives $\quad a_2 = -\dfrac{a_1}{10} = \dfrac{a_0}{3 \times 10}$

and from the general relationship, (4), we have ....

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$$\boxed{a_{r+1} = \frac{-a_r}{(r+1)(2r+3)}}$$

So $\quad a_1 = -\dfrac{a_0}{3}$

$$a_2 = -\frac{a_1}{2 \times 5} = \frac{a_0}{(1 \times 2)(3 \times 5)}$$

$$a_3 = -\frac{a_2}{3 \times 7} = \frac{-a_0}{(1 \times 2 \times 3)(3 \times 5 \times 7)}$$

$$a_4 = -\frac{a_3}{4 \times 9} = \frac{a_0}{(1 \times 2 \times 3 \times 4)(3 \times 5 \times 7 \times 9)} \quad \text{etc.}$$

$$y = x^c\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_r x^r + \ldots\}$$

i.e. $\quad y = \ldots\ldots\ldots\ldots$

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$$y = x^{\frac{1}{2}}\left\{a_0 - \frac{a_0}{3}x + \frac{a_0}{(1\times 2)(3\times 5)}x^2 - \frac{a_0}{(1\times 2\times 3)(3\times 5\times 7)}x^3 + \ldots\right\}$$

i.e. $$y = a_0 x^{\frac{1}{2}}\left\{1 - \frac{x}{(1\times 3)} + \frac{x^2}{(1\times 2)(3\times 5)} - \frac{x^3}{(1\times 2\times 3)(3\times 5\times 7)} + \ldots\right\}$$

Since a_0 is an arbitrary (non-zero) constant in each solution, its values may well be different, A and B say. If we denote the first solution by $u(x)$ and the second by $v(x)$, then

$$u = A\left\{1 - x + \frac{x^2}{(2\times 3)} - \frac{x^3}{(2\times 3)(3\times 5)} + \frac{x^4}{(2\times 3\times 4)(3\times 5\times 7)} + \ldots\right\}$$

and

$$v = Bx^{\frac{1}{2}}\left\{1 - \frac{x}{(1\times 3)} + \frac{x^2}{(1\times 2)(3\times 5)} - \frac{x^3}{(1\times 2\times 3)(3\times 5\times 7)} + \ldots\right\}$$

The general solution $y = u + v$ is therefore ....

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$$y = A\left\{1 - x + \frac{x^2}{(2\times 3)} - \frac{x^3}{(2\times 3)(3\times 5)} + \ldots\right\} + Bx^{\frac{1}{2}}\left\{1 - \frac{x}{(1\times 3)} + \frac{x^2}{(1\times 2)(3\times 5)} - \frac{x^3}{(1\times 2\times 3)(3\times 5\times 7)} + \ldots\right\}$$

The method may seem somewhat lengthy, but we have set it out in detail. It is a straightforward routine. Here is another example with the same steps.

Example 2

Find the series solution for the equation

$$3x^2y'' - xy' + y - xy = 0.$$

We proceed in just the same way as in the previous example.

Assume $\quad y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots\}$

i.e. $\quad y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \ldots + a_rx^{c+r} + \ldots$

$$\therefore\ y' = a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \ldots + a_r(c+r)x^{c+r-1} + \ldots$$

and $\quad y'' = \ldots\ldots\ldots\ldots$

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$$y'' = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + a_2(c+2)(c+1)x^c + \dots + a_r(c+r)(c+r-1)x^{c+r-2} + \dots$$

Now we build up the terms in the given equation.

$$3x^2y'' = 3a_0c(c-1)x^c + 3a_1(c+1)cx^{c+1} + 3a_2(c+2)(c+1)x^{c+2} + \dots + 3a_r(c+r)(c+r-1)x^{c+r} + \dots$$

$$-xy' = -a_0cx^c - a_1(c+1)x^{c+1} - a_2(c+2)x^{c+2} - \dots - a_r(c+r)x^{c+r} - \dots$$

$$y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \dots + a_rx^{c+r} + \dots$$

$$-xy = -a_0x^{c+1} - a_1x^{c+2} - \dots - a_rx^{c+r+1}\dots$$

The *indicial equation*, i.e. equating the coefficient of the lowest power of x to zero, gives the values of c. Thus, in this case

$c = \dots\dots\dots\dots$

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$$c = 1 \text{ or } \frac{1}{3}$$

Because the lowest power is x^c and the coefficient of x^c equated to zero gives

$$3a_0c(c-1) - a_0c + a_0 = 0$$

$$\therefore\ a_0(3c^2 - 4c + 1) = 0 \quad \therefore\ (3c-1)(c-1) = 0 \text{ since } a_0 \neq 0$$

$$\therefore\ c = 1 \text{ or } \frac{1}{3}$$

The coefficient of the general term, i.e. x^{c+r} gives

$$3a_r(c+r)(c+r-1) - a_r(c+r) + a_r - a_{r-1} = 0$$

$\therefore\ a_r = \dots\dots\dots\dots$

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$$a_r = \frac{a_{r-1}}{3(c+r)^2 - 4(c+r) + 1} = \frac{a_{r-1}}{(c+r-1)(3c+3r-1)}$$

(a) *Using* $c = 1$ the recurrence relation becomes

$$a_r = \frac{a_{r-1}}{r(3r+2)}$$

$$\therefore\ r = 1 \qquad a_1 = \frac{a_0}{1 \times 5}$$

$$r = 2 \qquad a_2 = \frac{a_1}{2 \times 8} = \frac{a_0}{(1 \times 2)(5 \times 8)}$$

$$r = 3 \qquad a_3 = \frac{a_2}{3 \times 11} = \frac{a_0}{(1 \times 2 \times 3)(5 \times 8 \times 11)}$$

Our first solution is therefore

$y = \dots\dots\dots\dots$

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$$y = x^1\left\{a_0 + \frac{a_0 x}{(1\times 5)} + \frac{a_0 x^2}{(1\times 2)(5\times 8)} + \frac{a_0 x^3}{(1\times 2\times 3)(5\times 8\times 11)} + \dots\right\}$$

$$\therefore\ y = Ax\left\{1 + \frac{x}{(1\times 5)} + \frac{x^2}{(1\times 2)(5\times 8)} + \frac{x^3}{(1\times 2\times 3)(5\times 8\times 11)} + \dots\right\}$$

(b) *For the second solution*, we put $c = \frac{1}{3}$. The recurrence relation then becomes

$$a_r = \dots\dots\dots\dots$$

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$$a_r = \frac{a_{r-1}}{r(3r-2)}$$

Therefore we can now determine the coefficients for $r = 1, 2, 3, \dots$ and complete the second solution.

$$y = \dots\dots\dots\dots$$

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$$y = Bx^{\frac{1}{3}}\left\{1 + x + \frac{x^2}{(2\times 4)} + \frac{x^3}{(2\times 3)(4\times 7)} + \frac{x^4}{(2\times 3\times 4)(4\times 7\times 10)} + \dots\right\}$$

Because

$$a_1 = \frac{a_0}{1\times 1}; \qquad a_2 = \frac{a_1}{2\times 4} = \frac{a_0}{(1\times 2)(2\times 4)}$$

$$a_3 = \frac{a_2}{3\times 7} = \frac{a_0}{(2\times 3)(4\times 7)}$$

$$a_4 = \frac{a_3}{4\times 10} = \frac{a_0}{(2\times 3\times 4)(4\times 7\times 10)}$$

$$\therefore\ y = a_0 x^{\frac{1}{3}}\left\{1 + x + \frac{x^2}{(2\times 4)} + \frac{x^3}{(2\times 3)(4\times 7)} + \frac{x^4}{(2\times 3\times 4)(4\times 7\times 10)} + \dots\right\}$$

Therefore, the general solution is

$$y = \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} y &= Ax\left\{1 + \frac{x}{(1 \times 5)} + \frac{x^2}{(1 \times 2)(5 \times 8)} + \frac{x^3}{(1 \times 2 \times 3)(5 \times 8 \times 11)} + \dots\right\} \\ &+ Bx^{\frac{1}{3}}\left\{1 + x + \frac{x^2}{(2 \times 4)} + \frac{x^3}{(2 \times 3)(4 \times 7)} + \frac{x^4}{(2 \times 3 \times 4)(4 \times 7 \times 10)} + \dots\right\}\end{aligned}}$$

Example 3

Find the series solution for the equation

$$\frac{d^2y}{dx^2} - y = 0 \quad \text{i.e.} \quad y'' - y = 0.$$

As usual, we start off with the assumed solution

$$y = x^c\{a_0 + a_1x + a_2x^2 + \dots + a_rx^r + \dots\}$$

$$\text{i.e.} \quad y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \dots + a_rx^{c+r} + \dots$$

$$\therefore \quad y' = a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \dots + a_r(c+r)x^{c+r-1} + \dots$$

$$y'' = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + a_2(c+2)(c+1)x^c + \dots + a_r(c+r)(c+r-1)x^{c+r-2} + \dots$$

These three expansions are required regularly, so make a note of them

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Now we build up the terms in the left-hand side of the equation.

$$y'' = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + a_2(c+2)(c+1)x^c + \dots + a_r(c+r)(c+r-1)x^{c+r-2} + \dots$$

$$y = a_0x^c + a_1x^{c+1} + \dots + a_rx^{c+r} + \dots$$

The term in x^{c+r} in the first of these expansions is

....

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$$\boxed{a_{r+2}(c+r+2)(c+r+1)x^{c+r}}$$

Because replacing r by $(r+2)$ in $a_r(c+r)(c+r+1)x^{c+r-2}$ gives this result.

Then $y'' - y = \dots\dots\dots\dots$

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$$y'' - y = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + [a_2(c+2)(c+1) - a_0]x^c + \ldots + [a_{r+2}(c+r+2)(c+r+1) - a_r]x^{c+r} + \ldots$$

We now equate each coefficient in turn to zero, since the right-hand side of the equation is zero. The coefficient of the lowest power of x gives the *indicial equation* from which we obtain the values of c.

So, in this case, $c = \ldots\ldots\ldots\ldots$

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$$c = 0 \quad \text{or} \quad 1$$

For the term in x^{c-1}, we have

$[x^{c-1}]$: $\quad a_1(c+1)c = 0.$

With $c = 1$, $a_1 = 0$.

But with $c = 0$, a_1 is indeterminate, because any value of a_1 combined with the zero value of c would make the product zero.

$[x^c]$: $\quad a_2(c+2)(c+1) - a_0 = 0 \quad \therefore \; a_2 = \dfrac{a_0}{(c+1)(c+2)}$

For the general term

$[x^{c+r}]$: $\quad \ldots\ldots\ldots\ldots$

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$$a_{r+2} = \frac{a_r}{(c+r+1)(c+r+2)}$$

Because $a_{r+2}(c+r+2)(c+r+1) - a_r = 0$. Hence the result above.

From the indicial equation, $c = 0$ or $c = 1$.

(a) When $c = 0$ $\quad a_1$ is indeterminate

$$a_2 = \frac{a_0}{2}$$

In general $\quad a_{r+2} = \dfrac{a_r}{(r+1)(r+2)}$

$r = 1 \quad \therefore \; a_3 = \dfrac{a_1}{2 \times 3}$

$r = 2 \quad a_4 = \dfrac{a_2}{3 \times 4} = \dfrac{a_0}{4!}$

Therefore, one solution is $\ldots\ldots\ldots\ldots$

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$$\boxed{y = x^0\left\{a_0 + a_1x + \frac{a_0}{2!}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 \dots\right\}}$$

i.e. $y = a_0\left\{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right\} + a_1\left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right\}$

a_0 and a_1 are arbitrary constants depending on the boundary conditions.

$$\therefore\ y = A\left\{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right\} + B\left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right\}$$

Notice that these two series are the Maclaurin series expansions of the hyperbolic functions, so that

$$y = A\cosh x + B\sinh x$$

It is not very often the case that the series solution is so easily expressible in terms of known functions.

(b) Similarly,

when $c = 1$

$$a_1 = 0$$
$$a_2 = \frac{a_0}{2 \times 3}$$
$$a_{r+2} = \dots\dots\dots\dots$$

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$$\boxed{a_{r+2} = \frac{a_r}{(r+2)(r+3)}}$$

$$\therefore\ a_1 = 0$$
$$a_2 = \frac{a_0}{3!}$$

$r = 1$ $\quad a_3 = \frac{a_1}{3 \times 4} = 0$

$r = 2$ $\quad a_4 = \frac{a_2}{4 \times 5} = \frac{a_0}{5!}$

$r = 3$ $\quad a_5 = \frac{a_3}{5 \times 6} = 0$ etc.

A second solution with $c = 1$ is therefore

$$y = \dots\dots\dots\dots$$

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$$\boxed{y = a_0\left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right\}}$$

and, because a_0 is an arbitrary constant

$$y = C\left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots\right\}$$

Note: This is not, in fact, a separate solution, since it already forms the second series in the solution for $c = 0$ obtained previously. Therefore, the first solution, with its two arbitrary constants, A and B, gives the general solution. This happens when the two values of c differ by an integer.

Make a note of the following:

If the two values of c, i.e. c_1 and c_2, differ by an integer, and if $c = c_1$ results in a_1 being indeterminate, then this value of c gives the general solution.

The solution resulting from $c = c_2$ is then merely a multiple of one of the series forming the first solution.

Our last problem was an example of this.
So far, we have met two distinct cases concerning the two roots $c = c_1$ and $c = c_2$ of the indicial equation.

(a) *If c_1 and c_2 differ by a quantity NOT an integer* then two independent solutions, $y = u(x)$ and $y = v(x)$, are obtained. The general solution is then $y = Au + Bv$.

(b) *If c_1 and c_2 differ by an integer*, i.e. $c_2 = c_1 + n$, and if one coefficient (a_r) is indeterminate when $c = c_1$, the complete general solution is given by using this value of c. Using $c = c_1 + n$ gives a series which is a simple multiple of one of the series in the first solution.

Make a note of these two points in your record book. Then move on

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There is a third category to be added to (a) and (b) above.

(c) If the roots $c = c_1$ and $c = c_1 + n$ of the indicial equation differ by an integer and one coefficient (a_r) becomes infinite when $c = c_1$, the series is rewritten with a_0 replaced by $k(c - c_1)$.

Putting $c = c_1$ in the rewritten series and that of its derivative with respect to c gives two independent solutions.

Add this to the previous two. Then we will see how it works in practice

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Example 4

Find the series solution of the equation

$xy'' + (2+x)y' - 2y = 0.$

Using $y = x^c(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots)$ and its first two derivatives, the expansions for

$$\begin{aligned} xy'' &= \ldots\ldots\ldots\ldots \\ 2y' &= \ldots\ldots\ldots\ldots \\ xy' &= \ldots\ldots\ldots\ldots \\ -2y &= \ldots\ldots\ldots\ldots \end{aligned}$$

Method as before.

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$$\begin{aligned} xy'' &= a_0c(c-1)x^{c-1} + a_1(c+1)cx^c + a_2(c+2)(c+1)x^{c+1} + \ldots \\ &\qquad + a_r(c+r)(c+r-1)x^{c+r-1} + \ldots \\ 2y' &= 2a_0cx^{c-1} + 2a_1(c+1)x^c + 2a_2(c+2)x^{c+1} + 2a_3(c+3)x^{c+2} \\ &\qquad + \ldots + 2a_r(c+r)x^{c+r-1} + \ldots \\ xy' &= a_0cx^c + a_1(c+1)x^{c+1} + a_2(c+2)c^{c+2} + \ldots \\ &\qquad + a_r(c+r)x^{c+r} + \ldots \\ -2y &= -2a_0x^c - 2a_1x^{c+1} - 2a_2x^{c+2} - 2a_3x^{c+3} - \ldots \\ &\qquad - 2a_rx^{c+r} - \ldots \end{aligned}$$

From which, the indicial equation is ....

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$$\boxed{a_0(c^2 + c) = 0}$$

i.e. equating the coefficient of the lowest power of x, (x^{c-1}), to zero.

$a_0 \neq 0 \qquad \therefore\ c = 0 \ \text{ or } \ -1$

Also, from the expansions, the total coefficient of x^c gives

$$a_1 = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_1 = \frac{-a_0(c-2)}{(c+1)(c+2)}}$$

From the terms in x^c, all four expansions are involved, so we can form the recurrence relation from the coefficient of x^{c+r}.

$$a_{r+1} = \ldots\ldots\ldots\ldots$$

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$$a_{r+1} = \frac{-a_r(c+r-2)}{(c+r+1)(c+r+2)}$$

Because

$$a_{r+1}(c+r+1)(c+r) + 2a_{r+1}(c+r+1) + a_r(c+r) - 2a_r = 0$$
$$a_{r+1}(c+r+1)(c+r+2) + a_r(c+r-2) = 0$$
$$\therefore\ a_{r+1} = \frac{-a_r(c+r-2)}{(c+r+1)(c+r+2)} \qquad r \geq 0$$
$$\therefore\ a_2 = \dots\dots\dots\dots$$

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$$a_2 = \frac{a_0(c-1)(c-2)}{(c+1)(c+2)^2(c+3)}$$

and, from the recurrence relation, when $r = 2$

$$a_3 = \dots\dots\dots\dots$$

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$$a_3 = \frac{-a_0c(c-1)(c-2)}{(c+1)(c+2)^2(c+3)^2(c+4)}$$

$$\therefore\ y = a_0x^c\left\{1 - \frac{c-2}{(c+1)(c+2)}x + \frac{(c-1)(c-2)}{(c+1)(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+1)(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$

From the indicial equation above, the values of c are 0 and -1.

Putting $c = 0$, we have one solution

$$y = u = \dots\dots\dots\dots$$

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$$y = u = a_0\left\{1 + x + \frac{x^2}{6}\right\}$$

Note that coefficients after the x^2 term are zero, because of the factor c in the numerator.

Putting $c = -1$, we soon find that $\dots\dots\dots\dots$

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coefficients become infinite, because of the factor $(c+1)$ in the denominator.

Therefore, we substitute $a_0 = k(c - c_1) = k(c - [-1]) = k(c+1)$.

$$\therefore\ y = k(c+1)x^c\left\{1 - \frac{c-2}{(c+1)(c+2)}x + \frac{(c-1)(c-2)}{(c+1)(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+1)(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$

$$= kx^c\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$

Now, putting $c = -1$:

$$y = \dots\dots\dots\dots$$

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$$y = kx^{-1}\left\{3x + 3x^2 + \frac{x^3}{2}\right\}$$

All subsequent terms are zero, since the numerators all contain a factor $(c+1)$.

$$\therefore\ y = v = \left\{3 + 3x + \frac{x^2}{2}\right\}$$

is a solution.

A solution is also given by $\dfrac{\partial y}{\partial c} = 0$.

So, starting from

$$y = kx^c\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$

$$\frac{\partial y}{\partial c} = kx^c \ln x\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$
$$+ kx^c\frac{\partial}{\partial c}\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \dots\right\}$$

▶

We now have to determine the partial derivative of each term.

$$\frac{\partial}{\partial c}(c+1) = 1$$
$$\frac{\partial}{\partial c}\left\{\frac{c-2}{c+2}\right\} = \dots\dots\dots\dots$$

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$$\boxed{\frac{\partial}{\partial c}\left\{\frac{c-2}{c+2}\right\} = \frac{4}{(c+2)^2}}$$

Now we have to differentiate $\dfrac{(c-1)(c-2)}{(c+2)^2(c+3)}$

Let $t = \dfrac{(c-1)(c-2)}{(c+2)^2(c+3)}$

$$\therefore \quad \ln t = \ln(c-1) + \ln(c-2) - 2\ln(c+2) - \ln(c+3)$$
$$\therefore \quad \frac{1}{t}\frac{\partial t}{\partial c} = \frac{1}{c-1} + \frac{1}{c-2} - \frac{2}{c+2} - \frac{1}{c+3}$$
$$\therefore \quad \frac{\partial t}{\partial c} = \frac{(c-1)(c-2)}{(c+2)^2(c+3)}\left\{\frac{1}{c-1} + \frac{1}{c-2} - \frac{2}{c+2} - \frac{1}{c+3}\right\}$$

$\therefore$ when $c = -1$, $\dfrac{\partial}{\partial c}(c+1) = 1$

$$\frac{\partial}{\partial c}\left\{\frac{c-2}{c+2}\right\} = 4$$
$$\frac{\partial}{\partial c}\left\{\frac{(c-1)(c-2)}{(c+2)^2(c+3)}\right\} = \dots\dots\dots\dots$$

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$$\boxed{-10}$$

Therefore, when $c = -1$:

$$\frac{\partial y}{\partial c} = kx^{-1}\ln x\left\{0 + 3x + 3x^2 + \frac{x^3}{2} + \dots\right\}$$
$$+ kx^{-1}\{1 - 4x - 10x^2 + \dots\}$$

$\therefore$ Another solution is

$$y = w = C\left\{\ln x\left(3 + 3x + \frac{x^2}{2} + \dots\right) + x^{-1}(1 - 4x - 10x^2 + \dots)\right\}$$

▶

Now we have a problem, for we seem to have three separate series solutions for a second-order differential equation.

(a) $y = u = A\left(1 + x + \dfrac{x^2}{6}\right)$

(b) $y = v = B\left(3 + 3x + \dfrac{x^2}{2}\right)$

(c) $y = w = C\left\{\ln x\left(3 + 3x + \dfrac{x^2}{2} + \ldots\right) + x^{-1}(1 - 4x - 10x^3 + \ldots)\right\}$

But (b) is clearly a simple multiple of (a) and thus not a distinct solution. So finally, we have just (a) and (c).

i.e. $\quad y = u = A\left(1 + x + \dfrac{x^2}{6}\right)$

and $\quad y = w = B\left\{\ln\, x\left(3 + 3x + \dfrac{x^2}{2} + \ldots\right) + x^{-1}(1 - 4x - 10x^3 + \ldots)\right\}$

The complete solution is then $y = u + w$

In general if $c_1 - c_2 = n$ where n is a non-zero integer the solution is of the form:

$$y = (1 + k\ln x)x^{c_1}\{a_0 + a_1x + a_2x^2 + \ldots\} + x^{c_2}\{b_0 + b_1x + b_2x^2 + \ldots\}$$

Finally we have just one more variation to the list in Frames 67 and 68, so move on

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Example 5

Solve the equation $xy'' + y' - xy = 0$.

Start off as before and build up expansions for the terms in the left-hand side of the equation.

$$xy'' = \ldots\ldots\ldots\ldots$$
$$y' = \ldots\ldots\ldots\ldots$$
$$-xy = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} xy'' &= a_0c(c-1)x^{c-1} + a_1(c+1)cx^c + a_2(c+2)(c+1)x^{c+1} + \ldots \\ &\qquad + a_r(c+r)(c+r-1)x^{c+r-1} + \ldots \\ y' &= a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \ldots \\ &\qquad + a_r(c+r)x^{c+r-1} + \ldots \\ -xy &= \qquad -a_0x^{c+1} - a_1x^{c+2} - \ldots \\ &\qquad - a_rx^{c+r+1} - \ldots \end{aligned}}$$

The indicial equation, therefore, gives $c = \ldots\ldots\ldots\ldots$

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$$\boxed{c = 0 \quad \text{(twice)}}$$

Because $a_0\,\{c(c-1)+c\} = 0 \quad a_0 \neq 0 \quad \therefore\ c^2 = 0 \quad \therefore\ c = 0$ (twice)

Coefficient of x^c gives ....

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$$\boxed{a_1 = 0}$$

$[x^c]$: $\quad a_1(c^2+c+c+1) = 0 \quad \therefore\ a_1(c+1)^2 = 0 \quad \therefore\ a_1 = 0$

$[x^{c+1}]$: This involves all three expansions and from this point, we can use the general recurrence relation.

$[x^{c+r-1}]$: $\quad a_r\{(c+r)(c+r-1)+(c+r)\} - a_{r-2} = 0$

$$\therefore\ a_r(c+r)^2 = a_{r-2} \quad \therefore\ a_r = \frac{a_{r-2}}{(c+r)^2}$$

$$\therefore\ y = \ldots\ldots\ldots\ldots$$

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$$\boxed{y = x^c\left\{a_0 + \frac{a_0}{(c+2)^2}x^2 + \frac{a_0}{(c+2)^2(c+4)^2}x^4 + \ldots\right\}}$$

i.e. $\quad y = a_0x^c\left\{1 + \dfrac{x^2}{(c+2)^2} + \dfrac{x^4}{(c+2)^2(c+4)^2} + \ldots\right\}$

$\therefore$ *When* $c = 0$

$$y = u = A\left\{1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} + \ldots\right\} \qquad (1)$$

▶

This is one solution. Another is given by $v = \dfrac{\partial y}{\partial c}$

$$\frac{\partial y}{\partial c} = a_0 x^c \ln x \left\{1 + \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots\right\}$$

$$+ a_0 x^c \frac{\partial}{\partial c}\left\{1 + \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots\right\}$$

Now $\dfrac{\partial}{\partial c}(1) = 0;\quad \dfrac{\partial}{\partial c}\left\{\dfrac{1}{(c+2)^2}\right\} = \dfrac{-2}{(c+2)^3}$

Let $t = \dfrac{1}{(c+2)^2(c+4)^2} \quad \therefore \quad \ln t = -2\ln(c+2) - 2\ln(c+4)$

$\therefore \quad \dfrac{1}{t}\dfrac{\partial t}{\partial c} = \dfrac{-2}{c+2} - \dfrac{2}{c+4} \quad \therefore \quad \dfrac{\partial t}{\partial c} = \dfrac{-2}{(c+2)^2(c+4)^2}\left\{\dfrac{1}{c+2} + \dfrac{1}{c+4}\right\}$

$$\therefore \quad \frac{\partial y}{\partial c} = a_0 x^c \ln x \left\{1 + \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots\right\}$$

$$+ a_0 x^c \left\{0 - \frac{2x^2}{(c+2)^3} - \frac{4x^4(c+3)}{(c+2)^3(c+4)^3} + \dots\right\}$$

$\therefore$ When $c = 0$

$$y = v = \dots\dots\dots\dots$$

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$$\boxed{y = v = B\left\{\ln x\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} + \dots\right) - \frac{x^2}{2^2} - \frac{3x^4}{2^3 \times 4^2} + \dots\right\}} \quad (2)$$

So our two solutions are $y = u$ (at 1) and $y = v$ (at 2). The complete solution is therefore $y = u + v$.

In general if $c_1 = c_2 = c$ the solution is of the form

$$y = (1 + k\ln x)x^c\{a_0 + a_1 x + a_2 x^2 + \dots\} + x^c\{b_1 x + b_2 x^2 + \dots\}$$

Summary

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Let us now summarise the four types of procedures in the method of Frobenius that we have covered.

(a) Assume a series of the form

$$y = x^c(a_0 + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots)$$

(b) Indicial equation gives $c = c_1$ and $c = c_2$.

(c) *Case 1.* c_1 and c_2 differ by a quantity *not an integer*. Substitute $c = c_1$ and $c = c_2$ in the series for y.

(d) *Case 2.* c_1 and c_2 differ by *an integer* and make a coefficient *indeterminate* with $c = c_1$. Substitution of $c = c_1$ gives the complete solution.

(e) *Case 3.* c_1 and c_2 $(c_1 < c_2)$ differ by *an integer* and make a coefficient *infinite* for $c = c_1$. Replace a_0 by $k\,(c - c_1)$. Put $c = c_1$ in the new series for y and for $\dfrac{\partial y}{\partial c}$.

In general if $c_1 - c_2 = n$ where n is a non-zero integer, the solution is of the form

$$y = (1 + k\ln x)x^{c_1}\{a_0 + a_1x + a_2x^2 + \ldots\} + x^{c_2}\{b_0 + b_1x + b_2x^2 + \ldots\}$$

(f) *Case 4.* c_1 and c_2 *equal*. Substitute $c = c_1$ in the series for y and for $\dfrac{\partial y}{\partial c}$. Make the substitution after differentiating.

In general if $c_1 = c_2 = c$, the solution is of the form

$$y = (1 + k\ln x)x^c\{a_0 + a_1x + a_2x^2 + \ldots\} + x^c\{b_1x + b_2x^2 + \ldots\}$$

Make a note of this summary for future reference

Bessel's equation

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A second-order differential equation that occurs frequently in branches of technology is of the form

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

where v is a real constant.

Starting with $y = x^c(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots)$ and proceeding as before, we obtain

$$c = \pm v \quad \text{and} \quad a_1 = 0$$

The recurrence relation is $\quad a_r = \dfrac{a_{r-2}}{v^2 - (c + r)^2}$ for $r \geq 2$.

It follows that $a_1 = a_3 = a_5 = a_7 = \ldots = 0$

and that $\quad a_2 = \ldots\ldots\ldots\ldots; \quad a_4 = \ldots\ldots\ldots\ldots; \quad a_6 = \ldots\ldots\ldots\ldots$

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$$a_2 = \frac{a_0}{v^2 - (c+2)^2}; \quad a_4 = \frac{a_0}{\left[v^2 - (c+2)^2\right]\left[v^2 - (c+4)^2\right]};$$
$$a_6 = \frac{a_0}{\left[v^2 - (c+2)^2\right]\left[v^2 - (c+4)^2\right]\left[v^2 - (c+6)^2\right]}$$

$\therefore$ *When* $c = +v$ $\quad a_2 = \ldots\ldots\ldots\ldots;\quad a_4 = \ldots\ldots\ldots\ldots$

$a_6 = \ldots\ldots\ldots\ldots;\quad a_r = \ldots\ldots\ldots\ldots$

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$$a_2 = \frac{-a_0}{2^2(v+1)}; \quad a_4 = \frac{a_0}{2^4 \times 2(v+1)(v+2)}$$
$$a_6 = \frac{-a_0}{2^6 \times 3!(v+1)(v+2)(v+3)}$$
$$a_r = \frac{(-1)^{r/2}a_0}{2^r \times (r/2)!(v+1)(v+2)\ldots(v+r/2)} \text{ for } r \text{ even}$$

The resulting series solution is therefore

$$y = u = \ldots\ldots\ldots\ldots$$

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$$y = u = Ax^v\left\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!(v+1)(v+2)(v+3)} + \ldots\right\}$$

This is valid provided v is not a negative integer.

Similarly, *when* $c = -v$

$$y = w = Bx^{-v}\left\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \ldots\right\}$$

This is valid provided v is not a positive integer.

Except for these two restrictions, the complete solution of Bessel's equation is therefore $y = u + w$ with the two arbitrary constants A and B.

Bessel functions

92

It is convenient to present the two results obtained above in terms of gamma functions, remembering that for $x > 0$

$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(x+2) = (x+1)\Gamma(x+1) = (x+1)x\Gamma(x)$$
$$\Gamma(x+3) = (x+2)\Gamma(x+2) = (x+2)(x+1)x\Gamma(x), \quad \text{etc.}$$

If, at the same time, we assign to the arbitrary constant a_0 the value $\dfrac{1}{2^{\nu}\Gamma(\nu+1)}$, then we have, for $c = \nu$

$$a_2 = \frac{a_0}{\nu^2 - (c+2)^2} = \frac{a_0}{(\nu - c - 2)(\nu + c + 2)} = \frac{a_0}{-2(2\nu+2)}$$
$$= \frac{-1}{2^2(\nu+1)} \cdot \frac{1}{2^{\nu}\Gamma(\nu+1)} = \frac{-1}{2^{\nu+2}(1!)\Gamma(\nu+2)}$$

Similarly $\qquad a_4 = \ldots\ldots\ldots\ldots$

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$$\boxed{a_4 = \frac{1}{2^{\nu+4}(2!)\Gamma(\nu+3)}}$$

Because

$$a_4 = \frac{a_2}{\nu^2 - (c+4)^2} = \frac{a_2}{(\nu - c - 4)(\nu + c + 4)} = \frac{a_2}{-4(2\nu+4)}$$
$$= \frac{-1}{2^3(\nu+2)} \cdot \frac{-1}{2^{\nu+2}(1!)\Gamma(\nu+2)} = \frac{1}{2^{\nu+4}(2!)\Gamma(\nu+3)}$$

and $a_6 = \ldots\ldots\ldots\ldots$

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$$\boxed{a_6 = \frac{-1}{2^{\nu+6}(3!)\Gamma(\nu+4)}}$$

We can see the pattern taking shape.

$$a_r = \frac{(-1)^{r/2}}{2^{\nu+r}\left(\frac{r}{2}!\right)\Gamma\left(\nu+\frac{r}{2}+1\right)} \quad \text{for } r \text{ even.} \qquad \therefore \text{ Put } r = 2k$$

The result then becomes

$$a_{2k} = \ldots\ldots\ldots\ldots$$

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$$a_{2k} = \frac{(-1)^k}{2^{v+2k}(k!)\Gamma(v+k+1)} \qquad k = 1, 2, 3, \ldots$$

Therefore, we can write the new form of the series for y as

$$y = x^v\left\{\frac{1}{2^v\Gamma(v+1)} - \frac{x^2}{2^{v+2}(1!)\Gamma(v+2)} + \frac{x^4}{2^{v+4}(2!)\Gamma(v+3)} - \ldots\right\}$$

This is called the *Bessel function of the first kind of order v* and is denoted by $J_v(x)$.

$$\therefore\ J_v(x) = \left(\frac{x}{2}\right)^v\left\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+3)} - \ldots\right\}$$

This is valid provided v is not ....

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a negative integer

– otherwise some of the terms would become infinite.

If we take the other value for c, i.e. $c = -v$, the corresponding result becomes

$$J_{-v}(x) = \ldots\ldots\ldots\ldots$$

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$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v}\left\{\frac{1}{\Gamma(1-v)} - \frac{x^2}{2(1!)\Gamma(2-v)} + \frac{x^4}{2^2(2!)\Gamma(3-v)} - \ldots\right\}$$

provided that v is not a positive integer.

In general terms

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(v+k+1)}$$

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-v+1)}$$

The convergence of the series for all values of x can be established by the normal ratio test.

$J_v(x)$ and $J_{-v}(x)$ are two independent solutions of the original equation. Hence, the complete solution is

$$y = AJ_v(x) + BJ_{-v}(x)$$

where A and B are constants.

Make a note of the expressions for $J_v(x)$ and $J_{-v}(x)$.
Then on to the next frame

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Some Bessel functions are commonly used and are worthy of special mention. This arises when v is a positive integer, denoted by n.

$$\therefore\ J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(n+k+1)}$$

From our work on gamma functions, $\Gamma(k+1) = k!$ for $k = 0, 1, 2, \ldots$

$$\therefore\ \Gamma(n+k+1) = (n+k)!$$

and the result above then becomes

$$J_n(x) = \ldots\ldots\ldots\ldots$$

99

$$\boxed{J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)(n+k)!}}$$

We have seen that $J_v(x)$ and $J_{-v}(x)$ are two solutions of Bessel's equation. When v and $-v$ are not integers, the two solutions are independent of each other. Then $y = AJ_v(x) + BJ_{-v}(x)$.

When, however, $v = n$ (integer), then $J_n(x)$ and $J_{-n}(x)$ are not independent, but are related by $J_{-n}(x) = (-1)^n J_n(x)$. This can be shown by referring once again to our knowledge of gamma functions.

$$\Gamma(x+1) = x\Gamma(x) \quad \therefore\ \Gamma(x) = \frac{\Gamma(x+1)}{x}$$

and for negative integral values of x, or zero, $\Gamma(x)$ is infinite.

From the previous result:

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-v+1)} \qquad k = 0, 1, 2, \ldots$$

Let us consider the gamma function $\Gamma(k-v+1)$ in the denominator and let v approach closely to a positive integer n.

Then $\quad \Gamma(k-v+1) \to \Gamma(k-n+1)$.

When $k-n+1 \leq 0$, i.e. when $k \leq (n-1)$, then $\Gamma(k-n+1)$ is infinite.

The first finite value of $\Gamma(k-n+1)$ occurs for $k = n$.

When values of $\Gamma(k-v+1)$ are infinite the coefficients of $J_{-v}(x)$ are

$\ldots\ldots\ldots\ldots$

100

$$\boxed{\text{zero}}$$

The series, therefore, starts at $k = n$

$$\therefore\ J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-n+1)}$$

$$= \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n}(k!)\Gamma(k-n+1)} \qquad \text{Put } k = p+n$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^{p+n} x^{2p+n}}{2^{2p+n}(k!)(k-n)!}$$

$$= (-1)^n \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p+n}}{2^{2p+n}(p!)(p+n)!}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p}}{2^{2p}(p!)(p+n)!}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)(k+n)!}$$

$$\therefore\ J_{-n}(x) = (-1)^n J_n(x)$$

So, after all that, the series for $J_n(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \ldots\ldots\ldots\ldots\right\}}$$

From this we obtain two commonly used functions

$$J_0(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots}$$

and $\qquad J_1(x) = \ldots\ldots\ldots\ldots$

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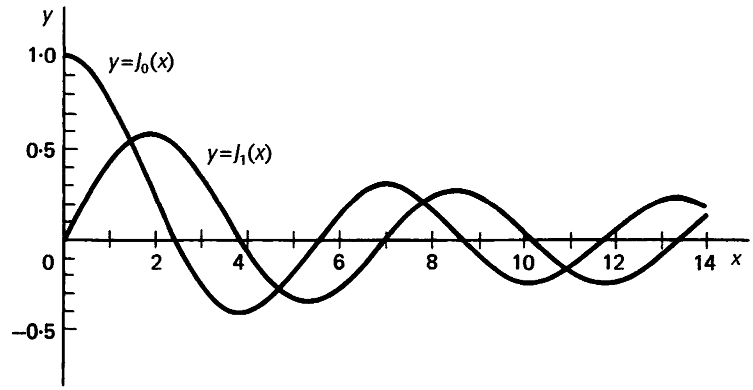
$$\boxed{J_1(x) = \frac{x}{2}\left\{1 - \frac{1}{(1!)(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 + \ldots\right\}}$$

Bessel functions for a range of values of n and x are tabulated in published lists of mathematical data. Of these, $J_0(x)$ and $J_1(x)$ are most commonly used.

Graphs of Bessel functions $J_0(x)$ and $J_1(x)$

104



Legendre's equation

105

Another equation of special interest in engineering applications is Legendre's equation of the form

$$(1 - x^2)y'' - 2xy' + k(k+1)y = 0$$

where k is a real constant.

This may be solved by the Frobenius method as before. In this case, the indicial equation gives $c = 0$ and $c = 1$, and the two corresponding solutions are

(a) $c = 0$: $\quad y = a_0\left\{1 - \dfrac{k(k+1)}{2!}x^2 + \dfrac{k(k-2)(k+1)(k+3)}{4!}x^4 - \ldots\right\}$

(b) $c = 1$: $\quad y = a_1\left\{x - \dfrac{(k-1)(k+2)}{3!}x^3 + \dfrac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5 - \ldots\right\}$

where a_0 and a_1 are the usual arbitrary constants

Legendre polynomials

When k is an integer (n), one of the solution series terminates after a finite number of terms. The resulting polynomial in x, denoted by $P_n(x)$, is called a *Legendre polynomial*, with a_0 or a_1 being chosen so that the polynomial has unit value when $x = 1$.

For example $\qquad P_2(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{P_2(x) = \frac{1}{2}(3x^2 - 1)}$$

Because, in $P_2(x)$, $\quad n = k = 2$

$$\therefore\ y = a_0\left\{1 - \frac{2 \times 3}{2!}x^2 + 0 + 0 + \ldots\right\}$$
$$= a_0\{1 - 3x^2\}$$

The constant a_0 is then chosen to make $y = 1$ when $x = 1$

i.e. $\quad 1 = a_0(1 - 3) \qquad \therefore\ a_0 = -\frac{1}{2}$

$$\therefore\ P_2(x) = -\tfrac{1}{2}(1 - 3x^2) = \tfrac{1}{2}(3x^2 - 1)$$

Similarly $\qquad P_3(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$$

Here $n = k = 3$

$$\therefore\ y = a_1\left\{x - \frac{2 \times 5}{3!}x^3 + 0 + 0 + \ldots\right\}$$
$$= a_1\left\{x - \frac{5x^3}{3}\right\}$$

$y = 1$ when $x = 1 \quad \therefore\ a_1\left(1 - \dfrac{5}{3}\right) = 1 \quad \therefore\ a_1 = -\dfrac{3}{2}$

$$\therefore\ P_3(x) = -\frac{3}{2}\left(x - \frac{5x^3}{3}\right) = \frac{1}{2}(5x^3 - 3x)$$

108 Rodrigue's formula and the generating function

Legendre polynomials can be derived by using *Rodrigue's formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

so using this formula

$$P_4(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)}$$

Because

$$\begin{aligned}
P_4(x) &= \frac{1}{2^4 4!}\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^2 - 1)^4 \\
&= \frac{1}{384}\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1) \\
&= \frac{1}{384}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(8x^7 - 24x^5 + 24x^3 - 8x) \\
&= \frac{1}{384}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(56x^6 - 120x^4 + 72x^2 - 8) \\
&= \frac{1}{384}\frac{\mathrm{d}}{\mathrm{d}x}(336x^5 - 480x^3 + 144x) \\
&= \frac{1}{384}(1680x^4 - 1440x^2 + 144) \\
&= \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}$$

In addition to Rodrigue's formula, the function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad |t| < 1$$

is called the *generating function* for Legendre polynomials and can be used to obtain some of their properties. For example using this generating function we find that

$$P_n(1) = \ldots\ldots\ldots\ldots$$

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$$\boxed{P_n(1) = 1}$$

Because

When $x = 1$ the generating function becomes

$$\frac{1}{\sqrt{1-2t+t^2}} = \sum_{n=0}^{\infty} P_n(1)t^n, \qquad |t| < 1$$

Noting that $\dfrac{1}{\sqrt{1-2t+t^2}} = \dfrac{1}{\sqrt{(1-t)^2}} = \dfrac{1}{1-t} = (1-t)^{-1}$, the left-hand side is expanded by the binomial theorem to give

$$(1-t)^{-1} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=0}^{\infty} t^n.$$

Therefore $\displaystyle\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n$ and so $P_n(1) = 1$

By a similar reasoning

$$P_n(-1) = \ldots\ldots\ldots\ldots$$

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$$\boxed{P_n(-1) = (-1)^n}$$

Because

When $x = -1$ the generating function becomes

$$\frac{1}{\sqrt{1+2t+t^2}} = \sum_{n=0}^{\infty} P_n(-1)t^n$$

Noting that $\dfrac{1}{\sqrt{1+2t+t^2}} = \dfrac{1}{\sqrt{(1+t)^2}} = \dfrac{1}{1+t} = (1+t)^{-1}$, the left-hand side is expanded by the binomial theorem to give

$(1+t)^{-1} = 1 - t + t^2 - t^3 + \ldots = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^n$. Therefore

$\displaystyle\sum_{n=0}^{\infty} (-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1)t^n$ and so $P_n(-1) = (-1)^n$

Legendre's equation, whose solutions are expressed in terms of Legendre polynomials, is an example of a particular class of differential equations referred to as Sturm–Liouville systems. In the following frames we shall look at such systems more closely.

So on to the next frame

Sturm–Liouville systems

112

A boundary value problem that is described by a differential equation of the general form

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0 \quad \text{for} \quad a \le x \le b \text{ and } r(x) > 0$$

where the boundary conditions can be written in the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

is called a **Sturm–Liouville** system. Solutions of such a system are in the form of an infinite sequence of *eigenfunctions* y_n, each corresponding to an *eigenvalue* λ_n of the system for $n = 0, 1, 2, \ldots$.

For example, consider the differential equation

$$y'' + \lambda y = 0 \quad \text{for} \quad 0 \le x \le 5$$

where here, $a = 0$ and $b = 5$. Also

$$y(0) = 0 \quad \text{and} \quad y(5) = 0$$

By comparing this equation with the general form given above we can see that

$p(x) = \ldots\ldots\ldots\ldots$; $q(x) = \ldots\ldots\ldots\ldots$; $r(x) = \ldots\ldots\ldots\ldots$;
$\alpha_2 = \ldots\ldots\ldots\ldots$; $\beta_2 = \ldots\ldots\ldots\ldots$

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$$\boxed{p(x) = 1; \quad q(x) = 0; \quad r(x) = 1; \quad \alpha_2 = 0; \quad \beta_2 = 0}$$

Because

By performing the differentiation on the left-hand term of $(p(x)y')' + (q(x) + \lambda r(x))y = 0$ we find that the differential equation can be written as

$$p(x)y'' + p'(x)y' + (q(x) + \lambda r(x))y = 0$$

By inspection, comparing this form with the differential equation $y'' + \lambda y = 0$ it is easily seen that $p(x) = 1$, $q(x) = 0$, $r(x) = 1$ and comparing boundary conditions gives $\alpha_2 = 0$ and $\beta_2 = 0$.

To solve the equation $y'' + \lambda y = 0$ we use the auxiliary equation $m^2 + \lambda = 0$ which has solutions $m = \pm j\sqrt{\lambda}$ (refer to *Engineering Mathematics (Fifth Edition)*, page 1077). This means that the solution can be written in the form

$$y = A \sin \ldots\ldots\ldots\ldots + B \cos \ldots\ldots\ldots\ldots$$

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$$\boxed{y = A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x}$$

Because

When the solutions to the auxiliary equation are of the form $m = \alpha \pm j\beta$ the solution to the differential equation is of the form

$$y = e^{\alpha x}(A \sin \beta x + B \cos \beta x) \quad \text{and here} \quad \alpha = 0 \quad \text{and} \quad \beta = \sqrt{\lambda}$$

Applying the boundary condition $y(0) = 0$ then $B = \ldots\ldots\ldots\ldots$

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$$\boxed{B = 0}$$

Because

$y = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$ and so $y(0) = A\sin 0 + B\cos 0 = B = 0$. Therefore $y = A\sin\sqrt{\lambda}x$

Applying the boundary condition $y(5) = 0$ then

$$\lambda = \ldots\ldots\ldots\ldots$$

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$$\boxed{\lambda = \frac{n^2\pi^2}{25}}$$

Because

$y = A\sin\sqrt{\lambda}x$ therefore $y(5) = A\sin\sqrt{\lambda}5 = 0$. If $A = 0$ the solution reduces to the trivial solution $y = 0$. For a non-trivial solution $\sin\sqrt{\lambda}5 = 0$ and so $\sqrt{\lambda}5 = n\pi$, $n = 0, 1, 2, 3, \ldots$. This means that

$$\sqrt{\lambda} = \frac{n\pi}{5} \quad \text{and so} \quad \lambda = \frac{n^2\pi^2}{25}$$

There is an infinity of eigenvalues, the nth eigenvalue being denoted by λ_n where $\lambda_n = \dfrac{n^2\pi^2}{25}$ and to each eigenvalue there is an eigenvector solution $y_n = A_n \sin\dfrac{n\pi x}{5}$.

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Orthogonality

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$ and

$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0$$

then we say that the two functions are mutually **orthogonal**. If, on the other hand, a third function $w(x) > 0$ exists such that

$$\int_a^b f(x)g(x)w(x)\,\mathrm{d}x = 0$$

then we say that $f(x)$ and $g(x)$ are mutually orthogonal *with respect to the weight function* $w(x)$.

One important property of the solutions to a Sturm–Liouville system is that the solutions are all mutually orthogonal with respect to the weight function $r(x)$. For instance, in the previous example the individual solutions were given as

$$y_n = A_n \sin\frac{n\pi x}{5} \quad \text{where} \quad r(x) = 1$$

and so if $m \ne n$

$$\int_0^5 y_m(x)y_n(x)r(x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_0^5 y_m(x)y_n(x)r(x)\,dx = 0}$$

Because

$$\begin{aligned}
\int_0^5 y_m(x)y_n(x)r(x)\,dx &= \int_0^5 A_m \sin\frac{m\pi x}{5} A_n \sin\frac{n\pi x}{5}\,dx \qquad \text{where } r(x) = 1\\
&= A_m A_n \int_0^5 \sin\frac{m\pi x}{5}\sin\frac{n\pi x}{5}\,dx\\
&= \frac{A_m A_n}{2}\int_0^5 \left(\cos\frac{(m-n)\pi x}{5} - \cos\frac{(m+n)\pi x}{5}\right)dx\\
&= \frac{A_m A_n}{2}\left[-\frac{5}{(m-n)\pi}\sin\frac{(m-n)\pi x}{5}\right.\\
&\qquad \left.+\frac{5}{(m+n)\pi}\sin\frac{(m+n)\pi x}{5}\right]_0^5 \quad \text{provided } m \neq n\\
&= 0
\end{aligned}$$

Summary

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1 A Sturm–Liouville system is a differential equation of the form

$$p(x)y'' + p'(x)y' + (q(x) + \lambda r(x))y = 0 \quad \text{for}$$
$$a \le x \le b \text{ and } r(x) > 0$$

where the boundary conditions can be written in the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

2 Solutions y_n to a Sturm–Liouville system are called eigenvectors, each corresponding to an eigenvalue λ_n for $n = 0, 1, 2, \ldots$.

3 The solutions y_n are mutually orthogonal with respect to the weighting $r(x)$. That is

$$\int_a^b y_m(x)y_n(x)r(x)\,dx = 0 \qquad (m \neq n)$$

Keep going

Legendre's equation revisited

120

The equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is Legendre's equation and has Legendre polynomials as solutions. That is

$$y_n = P_n(x) \qquad \text{where } P_n(1) = 1 \text{ and } P_n(-1) = (-1)^n$$

This equation is an example of a Sturm–Liouville system $p(x)y'' + p'(x)y' + (q(x) + \lambda r(x))y = 0$ with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0 \quad \text{where}$$

$$p(x) = \ldots\ldots\ldots\ldots; \quad q(x) = \ldots\ldots\ldots\ldots; \quad r(x) = \ldots\ldots\ldots\ldots;$$

$$\alpha_1, \alpha_2 = \ldots\ldots\ldots\ldots; \quad \beta_1, \beta_2 = \ldots\ldots\ldots\ldots$$

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$$p(x) = 1 - x^2; \quad q(x) = 0; \quad r(x) = 1; \quad \alpha_1, \alpha_2 = 1, 0; \quad \beta_1, \beta_2 = 1, 0$$

Consequently, Legendre polynomials are mutually orthogonal. That is, if $m \neq n$

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = 0}$$

Polynomials as a finite series of Legendre polynomials

Many differential equations cannot be solved by the normal analytical means and solution by power series provides a powerful tool in many situations. Furthermore, any polynomial can be written as a finite series of Legendre polynomials.

Example 1

Show that $f(x) = x^2$ can be written as a series of Legendre polynomials. Assume that

$$f(x) = x^2 = \sum_{n=0}^{\infty} a_n P_n(x), \text{ then}$$

$$\begin{aligned} x^2 &= a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \ldots \\ &= a_0(1) + a_1(x) + a_2\frac{3x^2-1}{2} + a_3\frac{5x^3-3x}{2} + \ldots \end{aligned}$$

Since the left-hand side is a polynomial of degree 2 then any Legendre polynomial on the right-hand side containing powers of x greater than 2 must be excluded. Therefore $a_3 = a_4 = \ldots = 0$, so that

$$x^2 = a_0 - \frac{a_2}{2} + a_1x + \frac{3}{2}a_2x^2 \quad \text{giving} \quad a_2 = \frac{2}{3},\ a_1 = 0,\ a_0 - \frac{a_2}{2} = 0$$

therefore $a_0 = \dfrac{1}{3}$, and

$$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$$

Now you try one

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Example 2

The polynomial $1 + x + x^3$ can be written as a series of Legendre polynomials in the form

$$1 + x + x^3 = \ldots\ldots\ldots\ldots$$

$$\boxed{1+x+x^3 = P_0(x) + \frac{8}{5}P_1(x) + \frac{2}{5}P_3(x)}$$ 124

Because

$$1+x+x^3 = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \ldots$$
$$= a_0(1) + a_1(x) + a_2\frac{3x^2-1}{2} + a_3\frac{5x^3-3x}{2} + \ldots$$

Since the left-hand side is a polynomial of degree 3 then any Legendre polynomial on the right-hand side containing powers of x greater than 3 must be excluded. Therefore $a_4 = a_5 = \ldots = 0$, so that

$$1+x+x^3 = a_0 - \frac{a_2}{2} + \left(a_1 - \frac{3}{2}a_3\right)x + \frac{3}{2}a_2x^2 + \frac{5}{2}a_3x^3$$

This gives $a_3 = \frac{2}{5}$, $a_2 = 0$, $a_1 - \frac{3}{2}a_3 = 1$, $a_0 - \frac{a_2}{2} = 1$ therefore $a_0 = 1$, and $a_1 = \frac{8}{5}$ so

$$1+x+x^3 = P_0(x) + \frac{8}{5}P_1(x) + \frac{2}{5}P_3(x)$$

As usual, the main points that we have covered in this Programme are listed in the **Revision summary** that follows. Read this in conjunction with the **Can You?** checklist and note any sections that may need further attention: refer back to the relevant parts of the Programme, if necessary. There will then be no trouble with the **Test exercise**. The set of **Further problems** provides an opportunity for further practice.

Revision summary 8 125

1 *Higher derivatives*

y	$y^{(n)}$
x^a	$\frac{a!}{(a-n)!}x^{a-n}$
e^{ax}	a^ne^{ax}
$\sin ax$	$a^n \sin\left(ax + \frac{n\pi}{2}\right)$
$\cos ax$	$a^n \cos\left(ax + \frac{n\pi}{2}\right)$
$\sinh ax$	$\frac{a^n}{2}\{[1+(-1)^n]\sinh ax + [1-(-1)^n]\cosh ax\}$
$\cosh ax$	$\frac{a^n}{2}\{[1-(-1)^n]\sinh ax + [1+(-1)^n]\cosh ax\}$

▶

2 *Leibnitz theorem* — nth derivative of a product of functions.

If $y = uv$

$$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \dots \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}u^{(n-r)}v^{(r)} + \dots$$

i.e. $y^{(n)} = \sum_{r=0}^{\infty} {}^{n}C_r u^{(n-r)}v^{(r)}$.

$(uv)^{(n)}$ can be obtained by expanding $(u+v)^{(n)}$ using the binomial theorem where the 'powers' are interpreted as derivatives.

3 *Power series solution of second-order differential equations*

(a) *Leibnitz–Maclaurin method*

(1) Differentiate the equation n times by the Leibnitz theorem.
(2) Put $x = 0$ to establish a recurrence relation.
(3) Substitute $n = 1, 2, 3, \dots$ to obtain $y', y'', y''', \dots$ at $x = 0$.
(4) Substitute in Maclaurin's series and simplify where possible.

(b) *Frobenius' method*

Assume a series solution of the form

$$y = x^c\{a_0 + a_1x + a_2x^2 + \dots + a_rx^r + \dots\} \quad a_0 \neq 0$$

(1) Differentiate the assumed series to find y' and y''.
(2) Substitute in the equation.
(3) Equate coefficients of corresponding powers of x on each side of the equation – usually written with zero on the right-hand side.
(4) Coefficient of the lowest power of x gives the *indicial equation* from which values of c are obtained, $c = c_1$ and $c = c_2$.

Case 1: c_1 and c_2 differ by a quantity *not an integer*. Substitute $c = c_1$ and $c = c_2$ in the series for y.

Case 2: c_1 and c_2 differ by an *integer* and make a coefficient *indeterminate* when $c = c_1$. Substitute $c = c_1$ to obtain the complete solution.

Case 3: c_1 and c_2 $(c_1 < c_2)$ differ by an *integer* and make a coefficient *infinite* when $c = c_1$. Replace a_0 by $k(c - c_1)$. Two independent solutions then obtained by putting $c = c_1$ in the new series for y and for $\dfrac{\partial y}{\partial c}$.

▶

In general if $c_1 - c_2 = n$ where n is a non-zero integer, the solution is of the form

$$y = (1 + k\ln x)x^{c_1}\{a_0 + a_1x + a_2x^2 + \dots\}$$
$$+ x^{c_2}\{b_0 + b_1x + b_2x^2 + \dots\}$$

Case 4: c_1 and c_2 are *equal*. Substitute $c = c_1$ in the series for y and for $\dfrac{\partial y}{\partial c}$. Make the substitution after differentiating. The second solution will consist of the product of the first solution and $\ln x$, together with a further series.

In general if $c_1 = c_2 = c$, the solution is of the form

$$y = (1 + k\ln x)x^{c}\{a_0 + a_1x + a_2x^2 + \dots\}$$
$$+ x^{c}\{b_1x + b_2x^2 + \dots\}$$

4 *Bessel's equation*

$$x^2y'' + xy' + (x^2 - v^2) = 0$$

where v is a real constant.

Bessel functions: Express the two solutions obtained in terms of gamma functions.

$$J_v(x) = \left(\frac{x}{2}\right)^v\left\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+3)} - \dots\right\}$$

This is the *Bessel function of the first kind of order v* – valid for v not a negative integer.

Also $J_{-v}(x) = \left(\dfrac{x}{2}\right)^{-v}\left\{\dfrac{1}{\Gamma(1-v)} - \dfrac{x^2}{2(1!)\Gamma(2-v)} + \dfrac{x^4}{2^2(2!)\Gamma(3-v)} - \dots\right\}$

provided that v is not a positive integer.

Complete solution is therefore $y = AJ_v(x) + BJ_{-v}(x)$.

When $v = n$ *(an integer)* $\quad J_{-n}(x) = (-1)^nJ_n(x)$

$$J_n(x) = \left(\frac{x}{2}\right)^n\left\{\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)(n+3)!}\left(\frac{x}{2}\right)^6 + \dots\right\}$$

In particular

$$J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \dots$$

and

$$J_1(x) = \frac{x}{2}\left\{1 - \frac{1}{(1!)(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)(4!)}\left(\frac{x}{2}\right)^6 + \dots\right\}$$

▶

5 *Legendre's equation*

$$(1-x^2)y''-2xy'+k(k+1)y=0$$

where k is a real constant.

Solution by Frobenius gives

$$c=0:\quad y=a_0\left\{1-\frac{k(k+1)}{2!}x^2+\frac{k(k-2)(k+1)(k+3)}{4!}x^4-\dots\right\}$$

$$c=1:\quad y=a_1\left\{x-\frac{(k-1)(k+2)}{3!}x^3\right.$$
$$\left.+\frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5-\dots\right\}$$

When k *is an integer*, one series terminates. The resulting polynomial in x, $P_n(x)$, is a *Legendre polynomial*, with a_0 or a_1 being chosen so that the polynomial has unit value when $x=1$.

6 *Rodrigue's formula*

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Generating function

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$$

7 *Sturm–Liouville systems*

$(p(x)y')'+(q(x)+\lambda r(x))y=0$ for $a\le x\le b$ and $r(x)>0$ with boundary conditions $\alpha_1 y(a)+\alpha_2 y'(a)=0$ and $\beta_1 y(b)+\beta_2 y(b)=0$

Solutions y_n to a Sturm–Liouville system are called eigenvectors, each corresponding to an eigenvalue λ_n for $n=0,1,2,\dots$

8 *Orthogonality*

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a\le x\le b$ and

$$\int_a^b f(x)g(x)\,dx=0$$

then the two functions are **orthogonal** to each other. If a function $w(x)>0$ exists such that

$$\int_a^b f(x)g(x)w(x)\,dx=0$$

then $f(x)$ and $g(x)$ are orthogonal to each other *with respect to the weight function* $w(x)$.

The solutions of a Sturm–Liouville system y_n are mutually orthogonal with respect to the weighting $r(x)$. That is

$$\int_a^b y_m(x)y_n(x)r(x)\,dx=0\qquad(m\ne n)$$

▶

9 *Legendre polynomials are mutually orthogonal*
If $m \neq n$ then

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

The orthogonality of the Legendre polynomials permits any polynomial to be written as a finite series of Legendre polynomials.

Can You?

Checklist 8

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	**Frames**
• Obtain the nth derivative of the exponential and circular and hyperbolic functions?	1 to 9
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply the Leibnitz theorem to derive the nth derivative of a product of expressions?	10 to 17
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply the Leibnitz–Maclaurin method of obtaining a series solution to a second-order homogeneous differential equation with constant coefficients?	18 to 36
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply Frobenius' method of obtaining a series solution to a second-order homogeneous differential equation for different cases of the indicial equation?	37 to 87
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply Frobenius' method to Bessel's equation to derive Bessel functions of the first kind?	88 to 104
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply Frobenius' method to Legendre's equation to derive Legendre polynomials?	104 to 107
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use Rodrigue's formula to derive Legendre polynomials and the generating function to obtain some of their properties?	108 to 111
Yes ☐ ☐ ☐ ☐ ☐ *No*	

▶

- Recognise a Sturm–Liouville system and the orthogonality properties of its eigenfunctions? 112 to 121

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Write a polynomial in x as a finite series of Legendre polynomials? 122 to 124

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 8

127

1 If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$.

2 Obtain a power series solution of the equation
$(1+x^2)y'' - 3xy' - 5y = 0$
up to and including the term in x^6.

3 Determine a series solution for each of the following.

(a) $3xy'' + 2y' + y = 0$

(b) $y'' + x^2y = 0$

(c) $xy'' + 3y' - y = 0$.

4 Use Rodrigue's formula $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ to derive the Legendre polynomials $P_2(x)$ and $P_3(x)$, and show that $P_2(x)$ and $P_3(x)$ are orthogonal on $(-1, 1)$.

5 Write $f(x) = 1 - 2x^2$ as a series of Legendre polynomials.

Further problems 8

128

(a) *Use the Leibnitz theorem* for the following.

1 If $y = x^3e^{4x}$, determine $y^{(5)}$.

2 Find the nth derivative of $y = x^3e^{-x}$ for $n > 3$.

3 If $y = x^3(2x+1)^2$, find $y^{(4)}$.

4 Find the 6th derivative of $y = x^4 \cos x$.

5 If $y = e^{-x}\sin x$, obtain an expression for $y^{(4)}$.

6 Determine $y^{(3)}$ when $y = x^4 \ln x$.

7 If $x^2y'' + xy' + y = 0$, show that

$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.$$

8 If $y = (2x - \pi)^4 \sin\left(\frac{x}{2}\right)$, evaluate $y^{(6)}$ when $x = \pi/2$.

9 If $y = e^{-x}\cos x$, show that $y^{(4)} + 4y = 0$.

10 Find the $(2n)$th derivative of (a) $y = x^2 \sinh x$
(b) $y = x^3 \cosh x$.

11 If $y = (x^3 + 3x^2)e^{2x}$, determine an expression for $y^{(6)}$.

12 Find the nth derivative of $y = e^{-ax}\cos ax$ and hence determine $y^{(3)}$.

13 If $y = \dfrac{\sin x}{1 - x^2}$, show that

(a) $(1 - x^2)y'' - 4xy' - (1 + x^2)y = 0$

(b) $y^{(n+2)} - (n^2 + 3n + 1)y^{(n)} - n(n-1)y^{(n-2)} = 0$ at $x = 0$.

(b) ***Use the Leibnitz–Maclaurin method*** to determine series solutions for the following.

14 $(1 + x^2)y'' + xy' - 9y = 0$.

15 $(x + 1)y'' + (x - 1)y' - 2y = 0$.

16 $(1 - x^2)y'' - 7xy' - 9y = 0$.

17 $(1 - x^2)y'' - 2xy' + 2y = 0$.

18 $xy'' + y' + 2xy = 0$.

(c) ***Use the method of Frobenius*** to obtain series solutions of the following.

19 $3xy'' + y' - y = 0$.

20 $y'' + y = 0$.

21 $y'' - xy = 0$.

22 $3xy'' + 4y' + y = 0$.

23 $y'' - xy' + y = 0$.

24 $xy'' - 3y' + y = 0$.

25 $xy'' + y' - 3y = 0$.

▶

26 Verify that $y'' + \lambda y = 0$ where $y'(0) = 0$ and $y(2) = 0$ is a Sturm–Liouville system. Find the eigenvalues and eigenfunctions of the system and prove that they are orthogonal in (0, 2).

27 Series solutions of the equation $y'' - 2xy' + 2ny = 0$ are known as Hermite polynomials, $H_n(x)$, where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Derive the first four Hermite polynomials and show that they are orthogonal with respect to the weight e^{-x^2} in $(-\infty, \infty)$.

28 Series solutions of the equation $xy'' + (1 - x)y' + ny = 0$ are known as Laguerre polynomials, $L_n(x)$, where

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

Derive the first four Laguerre polynomials and show that they are orthogonal with respect to the weight e^{-x} in $(0, \infty)$.

29 Given the generating function for Laguerre polynomials $L_n(x)$ as

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n$$

show that $L_n(0) = n!$

30 Given the generating function for Hermite polynomials $H_n(x)$ as

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

show that $H_{2n+1}(0) = 0$.

31 Given the generating function for Legendre polynomials $P_n(x)$ as

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

show that $P_{2n+1}(0) = 0$.